

Solutions to all Odd Problems of

**BASIC CALCULUS OF PLANETARY ORBITS AND
INTERPLANETARY FLIGHT**

Chapter 1. Solutions of Odd Problems

Problem 1.1. In Figure 1.33, let V represent Mercury instead of Venus. Let $\alpha = \angle VES$ be the maximum angle that Copernicus observed after measuring it many times. As in the case of Venus, α is at its maximum when the line of sight EV is tangent to the orbit of Mercury (assumed to be a circle). This means that when α is at its maximum, the triangle EVS is a right triangle with right angle at V . Since Copernicus concluded from his observations that $\frac{VS}{ES} \approx 0.38$, we know that $\sin \alpha = \frac{VS}{ES} \approx 0.38$. Working backward (today we would push the inverse sine button of a calculator) provides the conclusion that $\alpha \approx 22\frac{1}{3}^\circ$. So the maximum angle that Copernicus observed was approximately $22\frac{1}{3}^\circ$. From this he could conclude that $EV \approx 0.38$.

Problem 1.3. Since the lines of site BA and $B'A$ of Figure 1.36a are parallel, Figure 1.36b tells us that $(\alpha + \beta) + (\alpha' + \beta') = \pi$ as asserted. By a look at the triangle $\triangle BB'M$, $\beta + \beta' + \theta = \pi$. So $\beta + \beta' = \pi - \theta$, and hence $\alpha' + \alpha + (\pi - \theta) = \pi$. So $\theta = \alpha + \alpha'$.

Problem 1.5. Let R the radius of the circle of Figure 1.40b. Turn to Figure 1.40a, and notice that $\cos \varphi = \frac{R}{r_E}$. The law of cosines applied to the triangle of Figure 1.40b tells us that $(BB')^2 = R^2 + R^2 - 2R \cdot R \cos \theta = 2R^2(1 - \cos \theta)$. So $BB' = R\sqrt{2(1 - \cos \theta)}$. Since the point B rotates on the circle of Figure 1.40b at a constant rate, the ratio of $6\frac{1}{6}$ hours to 24 hours is the same as the ratio of θ expressed in degrees to 360° . So $\theta = \frac{6\frac{1}{6}}{24}360 = 92.5^\circ$, and

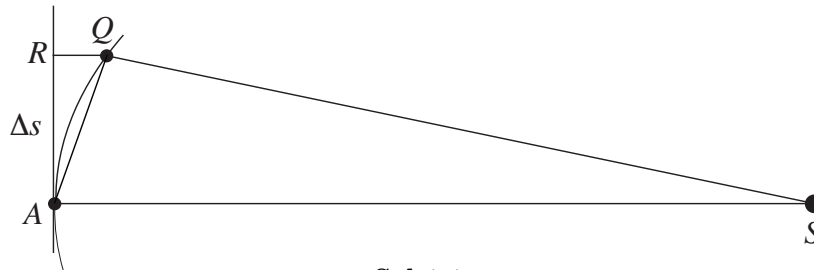
$$BB' = (r_E \cos \varphi) \sqrt{2(1 - \cos \theta)} = (6364.57 \cos 52.92^\circ) \sqrt{2(1 - \cos 92.5^\circ)} \approx 5543.98 \text{ km.}$$

Problem 1.7. Newton's computations of $\frac{a^3}{T^2}$ for Saturn's five moons, were

$$\frac{1.95^3}{45.31^2} \approx 0.00361, \quad \frac{2.5^3}{65.69^2} \approx 0.00362, \quad \frac{3.5^3}{108.42^2} \approx 0.00365, \quad \frac{8^3}{372.69^2} \approx 0.00369, \quad \text{and} \quad \frac{24^3}{1903.8^2} \approx 0.00381,$$

where for each of the five moons, a is the radius of the circular orbit (and hence also the semimajor axis of the orbit) measured in outer ring radii, and T is the corresponding period of the orbit in hours. Given the difficulty of determining the radii and the periods of the orbits of these five moons with precision with the instruments that existed in Newton's time, these results seem close enough to confirm Kepler's third law for Saturn and these moons.

Figure Sol 1.1 shows a planet (comet or asteroid) at its aphelion position A and a short stretch of its orbit QA before its arrival at A . Its time of travel from Q to A is Δt . It has already been



Sol 1.1

established that a planet (or comet or asteroid) attains its minimum speed v_{\min} at aphelion.

Problem 1.9. i. Let arc QA be the length of the arc between Q and A . Since Δt is the time of travel of the body, its average velocity v_{av} during its trip from Q to A is equal to $\frac{\text{arc } QA}{\Delta t}$. For a very small Δt , Q is close to A so that arc QA is close to Δs , so that $v_{av} = \frac{\text{arc } QA}{\Delta t} \approx \frac{\Delta s}{\Delta t}$.

ii. For a small Δs , the area ΔA of the elliptical sector SAQ is approximately equal to the area of the triangle ΔSAQ . Since the area of this triangle is $\frac{1}{2}(SA \cdot \Delta s)$ it follows that $\kappa = \frac{\Delta A}{\Delta t} \approx \frac{\frac{1}{2}(SA \cdot \Delta s)}{\Delta t}$. From Figure 1.9, $SA = a + c$, so that $\kappa \approx \frac{\frac{1}{2}(a+c) \cdot \Delta s}{\Delta t}$. Since $\kappa = \frac{ab\pi}{T}$ It follows that $\frac{\Delta s}{\Delta t} \approx \frac{2ab\pi}{(a+c)T}$.

iii. When Δt is pushed to zero, Q gets pushed to A , so that the average speed $v_{av} \approx \frac{\Delta s}{\Delta t}$, gets pushed to the speed v_{min} at aphelion. Simultaneously, the approximation of the area ΔA of the elliptical sector SAQ by the area $\frac{1}{2}(SA \cdot \Delta s) = \frac{1}{2}(a+c) \cdot \Delta s$ of ΔSAQ becomes tighter and tighter. Therefore, the approximations $\kappa = \frac{\Delta A}{\Delta t} \approx \frac{\frac{1}{2}(a+c) \cdot \Delta s}{\Delta t} \approx \frac{1}{2}(a+c)v_{min}$ snap to equalities. Since $\kappa = \frac{ab\pi}{T}$ this leads to the conclusion $v_{min} = \frac{2ab\pi}{(a+c)T}$.

iv. Since $c = \varepsilon a$ and $b = \sqrt{a^2 - c^2}$, we get $a + c = a(1 + \varepsilon)$ and $b = a\sqrt{1 - \varepsilon^2}$. So $v_{min} = \frac{2a^2\sqrt{1-\varepsilon^2}}{a(1+\varepsilon)} \frac{\pi}{T} = \frac{2a(\sqrt{1-\varepsilon})(\sqrt{1+\varepsilon})}{(\sqrt{1+\varepsilon})^2} \frac{\pi}{T} = \frac{2a\pi}{T} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$.

Recall that

$$1 \text{ au/year} = \frac{149,597,870.7 \text{ km}}{1 \text{ year}} \times \frac{1 \text{ year}}{31,557,600 \text{ sec}} \approx 4.74 \text{ km/sec.}$$

Problem 1.11. With Halley's semimajor axis $a = 17.83$ au, eccentricity $\varepsilon = 0.967$, and period $T = 75.32$ years, we get

$$v_{max} = \frac{2a\pi}{T} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} = \frac{2(17.83)\pi}{75.32} \sqrt{\frac{1+0.967}{1-0.967}} \approx 11.483 \text{ au/year} \approx 54.43 \text{ km/sec}$$

$$v_{min} = \frac{2a\pi}{T} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} = \frac{2(17.83)\pi}{75.32} \sqrt{\frac{1-0.967}{1+0.967}} \approx 0.193 \text{ au/year} \approx 0.91 \text{ km/sec.}$$

Problem 1.13. Using both $F = G \frac{MM_E}{d^2}$ and $F = \frac{4\pi^2 M}{T^2} \frac{M_E d}{M_E + M}$, we get $\frac{4\pi^2 M}{T^2} \frac{M_E d}{M_E + M} = G \frac{MM_E}{d^2}$, hence $\frac{4\pi^2}{T^2} \frac{d}{M_E + M} = \frac{G}{d^2}$, and therefore $\frac{4\pi^2}{T^2} \cdot \frac{d^3}{M_E + M} = G$. So $\frac{4\pi^2}{G} \cdot \frac{d^3}{T^2} = M_E + M$ and $\frac{4\pi^2}{GM_E} \cdot \frac{d^3}{T^2} = \frac{M_E + M}{M_E} = 1 + \frac{M}{M_E}$.

Problem 1.15. i. Let F be the force of gravity of Earth on the Moon, M_E the mass of Earth, and M that of the Moon. By Newton's law of universal gravitation,

$$F \approx \frac{GM_E M}{a_M^2} = \frac{(3.98600 \times 10^{14})(7.3461 \times 10^{22})}{(3.844 \times 10^8)^2} = 1.98165 \times 10^{20} \text{ newtons.}$$

ii. Let F be the force of gravity of the Sun on the Moon, M_S the mass of the Sun, and M that of the Moon. By Newton's law of universal gravitation,

$$F \approx \frac{GM_S M}{a_E^2} = \frac{(1.32712 \times 10^{20})(7.3461 \times 10^{22})}{(1.49598 \times 10^{11})^2} = 4.35628 \times 10^{20} \text{ newtons.}$$

iii. So the gravitational force of the Sun on the Moon is more than twice than that of Earth on the Moon. So why is the Moon orbiting Earth? The fact is that both Earth and Moon are orbiting the Sun. As these orbiting motions occur, the Earth pulls the Moon into orbit around it.

Problem 1.17. With $G = 6.67384 \times 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{sec}^2}$ in MKS, we get

$$G = 6.67384 \times 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{sec}^2} = 6.67384 \times 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{sec}^2} \cdot \frac{(100 \text{ cm})^3}{1 \text{ m}^3} \cdot \frac{1 \text{ kg}}{1000 \text{ gr}} \cdot \frac{1}{\text{sec}} = 6.67384 \times 10^{-8} \frac{\text{cm}^3}{\text{gr} \cdot \text{sec}^2}.$$

Problem 1.19. We know from section 1G that 1 foot = 30.48 cm and 1 inch = 2.54 cm. With the spheres of Figure 1.47 “distant but by $\frac{1}{4}$ inch”, $2d = \frac{2.54}{4}$ so that $d = \frac{2.54}{8} = 0.3175$ cm. Since the radius of the spheres is $\frac{1}{2}$ foot, $c = 15.24 + d = 15.24 + 0.3175 = 15.5575$ cm. The volume of each of the two spheres is $\frac{4}{3}\pi(15.24)^3 \text{ cm}^3 \approx 14827 \text{ cm}^3$. Under the assumption that their densities are the same as that of the Earth, the mass of each sphere is $(14827)(5.54) \text{ gr} = 82,142 \text{ gr}$.

Problem 1.21. If can determine the time t it takes for the mutual force of attraction to move each of the spheres through a distance d , then the two spheres will touch after time t and the problem is solved. The equality $x(t) = \frac{1}{2}\frac{F}{m}t^2$ informs us that $t^2 = \frac{2mx(t)}{F}$ and hence that $t = \sqrt{\frac{2mx(t)}{F}}$. With $x(t) = d = 0.3175$ cm and F equal to the magnitude of the maximal force of 0.485 dynes throughout the motion of the sphere, we get

$$t = \sqrt{\frac{2mx(t)}{F}} = \sqrt{\frac{2(82,142)(0.3175)}{0.485}} = 327.94 \text{ sec.}$$

Under the assumption that the minimal force acts throughout the motion, we get $t = \sqrt{\frac{2mx(t)}{F}} = \sqrt{\frac{2(82,142)(0.3175)}{0.465}} = 334.92 \text{ sec}$ and it takes a little longer for the two spheres to meet. Since the force of gravity between the two sphere varies from the smaller of these two values to the larger, the time it takes for the two spheres to meet lies between the two calculated values, thus about $5\frac{1}{2}$ minutes. This conclusion is of course at odds with Newton’s assertion about the matter. Why? While Newton could compute GM with M the Earth’s mass (from his version of Kepler’s third law), he had no clue about G (as he had asserted himself).

Problem 1.23. i. Let a and ε be the semimajor axis and eccentricity of *Sputnik 1*’s elliptical orbit. The addition of the distance 942 km to Earth’s radius of 6371 km gives an aphelion distance of $a + a\varepsilon = 7313$ km for the orbit, and adding 230 km to 6371 km gives a perihelion distance of $a - a\varepsilon = 6601$ km. So $2a = (a + \varepsilon a) + (a - \varepsilon a) = 7313 + 6601 = 13,914$ km and hence $a = 6957$ km. Since $2\varepsilon a = (a + \varepsilon a) - (a - \varepsilon a) = 7313 - 6601 = 712$ km, we get $a\varepsilon = 356$ km, and hence $\varepsilon = \frac{356}{6957} \approx 0.051$. *Sputnik 1*’s period is $T = 96.2 \cdot 60 = 5772$ in seconds. Using the speed formulas of Problems 1.8 and 1.9, we get

$$v_{\max} = \frac{2a\pi}{T} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} = \frac{2(6957)\pi}{5772} \sqrt{\frac{1+0.051}{1-0.051}} \approx 7.97 \text{ km/sec and}$$

$$v_{\min} = \frac{2a\pi}{T} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} = \frac{2(6957)\pi}{5772} \sqrt{\frac{1-0.051}{1+0.051}} \approx 7.20 \text{ km/sec.}$$

ii. Using Newton’s version of Kepler’s third law with M the Earth’s mass we get in MKS, $GM = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (6,957 \times 10^3)^3}{((96.2)(60))^2} \approx 3.990 \times 10^{14} \text{ m}^3/\text{sec}^2$.

Chapter 2. Solutions of Odd Problems

Problem 2.1. With M the mass of the Moon, Newton's version of Kepler's third law tells us that $GM = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (2413 \times 10^3)^3}{((178.05)(60))^2} \approx 4.8601 \times 10^{12} \frac{\text{m}^3}{\text{sec}^2}$. Taking $G = 6.674 \times 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{sec}^2}$ from Chapter 1H, we find that $M = 7.282 \times 10^{22} \text{ kg}$.

Problem 2.3. Using the data from the initial orbit (with MKS), we get

$$GM = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (13055 \times 10^3)^3}{((12.62)(3600))^2} \approx 4.2557 \times 10^{13} \text{ m}^3/\text{sec}^2.$$

For the post trim 1 orbit, $GM = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (12631 \times 10^3)^3}{((11.97)(3600))^2} \approx 4.2843 \times 10^{13} \text{ m}^3/\text{sec}^2$, and for the post trim 2 orbit, $GM = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (12647 \times 10^3)^3}{((11.99)(3600))^2} \approx 4.2863 \times 10^{13} \text{ m}^3/\text{sec}^2$.

Problem 2.5 refers to videos about the *Voyager* missions.

Problem 2.7. Let M be the mass of Mars in kg. Using the orbital data of Table 2.2 for the moon Phobos, we get $GM = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (9378 \times 10^3)^3}{((0.31891)(24 \cdot 60 \cdot 60))^2} \approx 4.28871 \times 10^{13} \frac{\text{m}^3}{\text{sec}^2}$. Using data for the orbit of Deimos, $GM = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (23459 \times 10^3)^3}{((1.26244)(24 \cdot 60 \cdot 60))^2} \approx 4.28390 \times 10^{13} \frac{\text{m}^3}{\text{sec}^2}$. After dividing both results by $G = 6.67384 \times 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{sec}^2}$, we get $M = 6.42615 \times 10^{23} \text{ kg}$ and $M = 6.41894 \times 10^{23} \text{ kg}$, respectively. (With regard to the "advertised" $M = 6.42409 \times 10^{23} \text{ kg}$ and $M = 6.39988 \times 10^{23} \text{ kg}$, it is safe to say that their computation made use of a different value of G and different roundoff strategies.)

Problem 2.9 is another invitation to explore interesting videos related to the topics of this chapter.

Problem 2.11. Kepler's third law and Newton's more explicit version of it tell us that $\frac{a^3}{T^2}$ is the same for any object in orbit around the Sun. Taking $a = 1 \text{ au}$ for Earth's semimajor axis and $T = 1 \text{ year}$ for its period, we get that in the units au and year, $\frac{a^3}{T^2} \approx 1$ for the Earth, and hence for any object in an elliptical orbit around the Sun. Therefore in these units $T^2 \approx a^3$. Recall the formula $v_{\max} = \frac{2a\pi}{T} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$ for the maximum speed of any object in an elliptical orbit in terms of its orbital data. Turning to the units au and year and noting that $\varepsilon < 1$, we get

$$v_{\max} = \frac{2a\pi}{T} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \approx \frac{2a\pi}{a^{\frac{3}{2}}} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \approx \frac{2\pi}{a^{\frac{1}{2}}} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \approx 2\pi \sqrt{\frac{1+\varepsilon}{a(1-\varepsilon)}} < \frac{2\pi \cdot \sqrt{2}}{\sqrt{a(1-\varepsilon)}},$$

where v_{\max} is expressed in au/year, and a and $a(1 - \varepsilon)$ are the semimajor axis and perihelion distance in au, respectively.

Problem 2.13. Table 2.8 tells us that the orbit of Great Comet of 1680 had an eccentricity very close to 2 and a perihelion distance of 0.0062 au. It follows from Problem 2.11 that the maximum speed that the Great Comet of 1680 attained was close to

$$v_{\max} \approx 2\pi \sqrt{\frac{1+\varepsilon}{a(1-\varepsilon)}} \approx \frac{2\pi \cdot \sqrt{2}}{\sqrt{a(1-\varepsilon)}} \approx \frac{2\pi \cdot \sqrt{2}}{\sqrt{0.0062}} \approx 112.85 \text{ au/year}$$

and hence $(112.85)(4.74) \approx 535 \text{ km/sec}$.

Problem 2.15. This computation is identical to that of the Great Comet of 1843, so that for Lovejoy's comet C/2011 W3,

$$v_{\max} \approx 2\pi \sqrt{\frac{1+\varepsilon}{a(1-\varepsilon)}} \approx \frac{2\pi \cdot \sqrt{2}}{\sqrt{a(1-\varepsilon)}} \approx \frac{2\pi \cdot \sqrt{2}}{\sqrt{0.0055}} \approx 119.82 \text{ au/year}$$

and hence $(119.82)(4.74) \approx 568 \text{ km/sec}$. The brief video

<https://phys.org/news/2012-03-comet-lovejoy-survive-sun.html>

that depicts its hurried scramble around the Sun is worth a look.

Chapter 3. Solutions of Odd Problems

Problem 3.1. Using the equalities $x = r \cos \theta$ and $y = r \sin \theta$, we get the Cartesian coordinates,

- i. Since $r = 0$, this is the origin $(0, 0)$.
- ii. $x = 5 \cos \frac{\pi}{6} \approx 4.33$ and $y = 5 \sin \frac{\pi}{6} = 2.5$.
- iii. $x = 7 \cos \frac{5\pi}{4} \approx -4.95$ and $y = 7 \sin \frac{5\pi}{4} \approx -4.95$.
- iv. $x = -6 \cos(-\frac{9\pi}{4}) \approx -4.24$ and $y = -6 \sin(-\frac{9\pi}{4}) \approx 4.24$.
- v. $x = 7 \cos 10 \approx -5.87$ and $y = 7 \sin 10 \approx -3.81$.
- vi. $x = -3 \cos(-20) \approx -1.22$ and $y = -3 \sin(-20) \approx -2.74$.

Problem 3.3. The polar equations are obtained by substituting the equalities $x = r \cos \theta$ and $y = r \sin \theta$ into the Cartesian equations.

- i. Since $2r \cos \theta + 3r \sin \theta = 4$, we get $r(2 \cos \theta + 3 \sin \theta) = 4$, and hence $r = \frac{4}{2 \cos \theta + 3 \sin \theta}$.
- ii. From $(r \cos \theta)^2 + (r \sin \theta)^2 = 4r \sin \theta$, we get $r^2 = 4r \sin \theta$, or either $r = 0$ or $r = 4 \sin \theta$. So the Cartesian equation is satisfied provided either of these two equations is satisfied. Since the polar origin $(0, 0)$ is on the graph of $r = 4 \sin \theta$, the equation $r = 4 \sin \theta$ is the relevant answer.
- iii. Since $(r \cos \theta)^2 + (r \sin \theta)^2 = r \cos \theta((r \cos \theta)^2 - 3(r \sin \theta)^2)$, we get $r^2 = r^3 \cos \theta(\cos^2 \theta - 3 \sin^2 \theta)$. So either $r = 0$ or $r = \frac{1}{\cos \theta(\cos^2 \theta - 3 \sin^2 \theta)}$. Note that the polar point $(0, 0)$ is not on the graph of this function.

Problem 3.5. The graphs turn out to be simple. Descriptions suffice.

- i. Since the graph of $r = -6$ consists of all polar points $(-6, \theta)$ for any θ , this is a circle of radius 6.
- ii. The graph of $\theta = -\frac{8\pi}{6}$ is the set of all polar points $(r, -\frac{8\pi}{6})$ for any r . So the graph is the entire line determined by the ray $\theta = -\frac{4\pi}{3}$. A moment's thought about the xy -plane tells us that this is the line through the origin at an angle of $\frac{\pi}{3} = 60^\circ$ with the positive x -axis.
- iii. Since the polar origin lies on the graph of $r = 4 \sin \theta$, the graph of this equation is the same as the graph of $r^2 = r \sin \theta$. In Cartesian coordinates this translates to $x^2 + y^2 = 4y$ and in turn to $x^2 + y^2 - 4y + 4 = 4$. Hence the graph is that of the Cartesian equation $x^2 + (y - 2)^2 = 4$. This is a circle with center the Cartesian point $(0, 2)$ and radius 2.
- iv. The Cartesian version of $r(\sin \theta + \cos \theta) = 1$ is $y + x = 1$ or $y = -x + 1$. This is the line with slope -1 through the Cartesian point $(0, 1)$.

Problem 3.7. With $x = r \cos \theta$ and $y = r \sin \theta$, the equation $ax + by + c = 0$ transforms to $r(a \cos \theta + b \sin \theta) = -c$. If $c \neq 0$, then $a \cos \theta + b \sin \theta \neq 0$, and $r = f(\theta) = \frac{-c}{a \cos \theta + b \sin \theta}$ is a polar function that has the line as its graph. (An ambitious reader might wish to determine a domain $?\ < \theta < ?$ for which the entire line is traced out.) If $c = 0$, the line is not the graph of a polar function. We can assume that $b \neq 0$ and consider the line $y = -\frac{a}{b}x$. Let θ_0 with $-\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}$ be the angle the line makes with the polar axis. Note that $\tan \theta_0 = \frac{y}{x} = -\frac{a}{b}$ is the slope of the line. For any point (r, θ) on the line (other than the origin), θ must be one of the angles $\theta_0, \theta_0 \pm \pi, \theta_0 \pm 2\pi, \dots$. Assume, if possible, that the line is the graph of a polar function $r = f(\theta)$. Since $\theta = \theta_0, \theta_0 \pm \pi, \theta_0 \pm 2\pi, \dots$, are the only angles that arise, the graph of $r = f(\theta)$ consists of the points, $(f(\theta_0), \theta_0), (f(\theta_0 + \pi), \theta_0 + \pi), (f(\theta_0 - \pi), \theta_0 - \pi), (f(\theta_0 + 2\pi), \theta_0 + 2\pi), (f(\theta_0 - 2\pi), \theta_0 - 2\pi), \dots$. But there are not enough of these to fill the whole line.

Problem 3.9. The solutions will rely on the discussion of section 3D and the equation $r = \frac{d}{1 + \varepsilon \cos \theta}$.

- i. By Figure 3.16, the directrix of the parabola is the vertical line $x = d$ and the focus is the origin. So $d = 10$ and $\varepsilon = 1$. Therefore the equation is $r = \frac{10}{1 + \cos \theta}$.
- ii. Since the semimajor and semiminor axes of the ellipse are $a = 8$ and $b = 5$ respectively, we know that $\frac{d}{1 - \varepsilon^2} = 8$ and $\frac{d}{\sqrt{1 - \varepsilon^2}} = 5$. Since $\frac{d^2}{1 - \varepsilon^2} = 25$, it follows that $d = \frac{d^2}{1 - \varepsilon^2} \cdot \frac{1 - \varepsilon^2}{d} = \frac{25}{8}$. Since $\frac{d}{1 - \varepsilon^2} = 8$, we get $1 - \varepsilon^2 = \frac{d}{8} = \frac{25}{64}$. So $\varepsilon^2 = \frac{39}{64}$ and $\varepsilon = \frac{\sqrt{39}}{8}$. Therefore the equation of the ellipse is $r = \frac{\frac{25}{8}}{1 + \frac{\sqrt{39}}{8} \cos \theta} = \frac{25}{8 + \sqrt{39} \cos \theta}$.
- iii. The fact that the semimajor and semiminor axes of the hyperbola are $a = 8$ and $b = 5$ respectively, tells us that $\frac{d}{\varepsilon^2 - 1} = 8$ and $\frac{d}{\sqrt{\varepsilon^2 - 1}} = 5$. Since $\frac{d^2}{\varepsilon^2 - 1} = 25$, we get $d = \frac{d^2}{\varepsilon^2 - 1} \cdot \frac{\varepsilon^2 - 1}{d} = \frac{25}{8}$. Since $\frac{d}{\varepsilon^2 - 1} = 8$, $\varepsilon^2 - 1 = \frac{d}{8} = \frac{25}{64}$. Therefore $\varepsilon^2 = \frac{89}{64}$ and $\varepsilon = \frac{\sqrt{89}}{8}$. So the equation of the hyperbola is $r = \frac{\frac{25}{8}}{1 + \frac{\sqrt{89}}{8} \cos \theta} = \frac{25}{8 + \sqrt{89} \cos \theta}$.

Problem 3.11. The ellipse discussed in section 3D part (ii) has semimajor axis $a = \frac{d}{1 - \varepsilon^2}$ and semiminor axis $b = \frac{d}{\sqrt{1 - \varepsilon^2}}$. Assuming that $a = 7$ and $b = 4$, we see that $\frac{d}{1 - \varepsilon^2} = 7$ and $\frac{d}{\sqrt{1 - \varepsilon^2}} = 4$. Therefore $\frac{d^2}{1 - \varepsilon^2} = 16$ and it follows that $d = \frac{d^2}{1 - \varepsilon^2} \cdot \frac{1 - \varepsilon^2}{d} = \frac{16}{7}$. Since $\frac{d}{1 - \varepsilon^2} = 7$, we see that $1 - \varepsilon^2 = \frac{d}{7} = \frac{16}{49}$. So $\varepsilon^2 = \frac{33}{49}$ and $\varepsilon = \frac{\sqrt{33}}{7}$. This means that the ellipse (*) given by $r = \frac{\frac{16}{7}}{1 + \frac{\sqrt{33}}{7} \cos \theta}$ has semimajor axis $a = 7$ and semiminor axis $b = 4$. The ellipse C and the ellipse (*) can each be moved in the plane to coincide with the ellipse that has equation $\frac{x^2}{7^2} + \frac{y^2}{4^2} = 1$. Therefore the ellipses C and (*) have the same shape.

Problem 3.13. The ellipse with equation $r = \frac{d}{1 + \varepsilon \cos \theta}$ and $\varepsilon < 1$ discussed in section 3D part (ii) has semimajor axis $a = \frac{d}{1 - \varepsilon^2}$ and semiminor axis $b = \frac{d}{\sqrt{1 - \varepsilon^2}}$. Since $b^2 = \frac{d^2}{1 - \varepsilon^2}$ we see that $d = \frac{d^2}{1 - \varepsilon^2} \cdot \frac{1 - \varepsilon^2}{d} = \frac{b^2}{a}$. Since $\frac{1 - \varepsilon^2}{d} = \frac{1}{a}$, we get $1 - \varepsilon^2 = \frac{d}{a} = \frac{b^2}{a^2}$. Hence $\varepsilon^2 = 1 - \frac{b^2}{a^2} = \frac{a^2 - b^2}{a^2}$ and $\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}$. It follows that the graph of the equation $r = \frac{\frac{b^2}{a}}{1 + \frac{\sqrt{a^2 - b^2}}{a} \cos \theta}$ is an ellipse with semimajor axis a and semiminor axis b . So the ellipse C and the ellipse (*) can each be moved in the plane to coincide

with the ellipse that has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore the ellipses C and $(*)$ have the same shape.

Problem 3.15. The equation $r = \frac{d}{1+\varepsilon \sin \theta}$, where $d > 0$ and $\varepsilon \geq 0$, can be written as $r + \varepsilon r \sin \theta = d$. The corresponding Cartesian equation is $\pm \sqrt{x^2 + y^2} + \varepsilon y = d$.

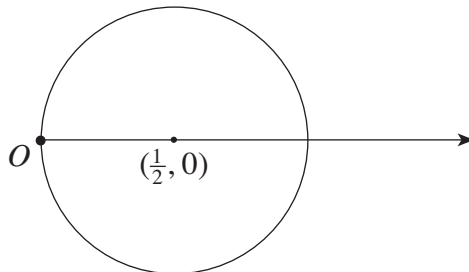
Let's begin with the case $\varepsilon = 1$. Since $\sqrt{x^2 + y^2} \geq y$ and $d > 0$, it follows that the minus alternative does not occur and hence that $\sqrt{x^2 + y^2} + y = d$. This is an equation of the parabola with focal point the origin O and directrix the horizontal line $y = d$. This can be seen as follows. Let $P = (x, y)$ be any point on the parabola with focal point the origin and directrix the horizontal line $y = d$. Its distance from the origin is $\sqrt{x^2 + y^2}$. Since $d > 0$ the directrix lies above the focal point. It follows that $y < d$ and that the distance from P to the directrix is $d - y$. So $\sqrt{x^2 + y^2} = d - y$ and hence $\sqrt{x^2 + y^2} + y = d$. Since this is identical to the earlier equation, it follows that the graph of $r = \frac{d}{1+\sin \theta}$ is a parabola with focal point the origin O and directrix the line $y = d$.

We'll now suppose that $\varepsilon \neq 1$. Squaring both sides of $\pm \sqrt{x^2 + y^2} = d - \varepsilon y$, we get in successive steps (one of them a completion of a square) that

$$\begin{aligned} x^2 + y^2 &= d^2 - 2\varepsilon dy + \varepsilon^2 y^2 \\ x^2 + (1 - \varepsilon^2)y^2 + 2\varepsilon dy &= d^2 \\ \frac{x^2}{1 - \varepsilon^2} + y^2 + \frac{2\varepsilon d}{1 - \varepsilon^2}y &= \frac{d^2}{1 - \varepsilon^2} \\ \frac{x^2}{1 - \varepsilon^2} + y^2 + \frac{2\varepsilon d}{1 - \varepsilon^2}y + \frac{\varepsilon^2 d^2}{(1 - \varepsilon^2)^2} &= \frac{d^2}{1 - \varepsilon^2} + \frac{\varepsilon^2 d^2}{(1 - \varepsilon^2)^2} \\ \frac{x^2}{1 - \varepsilon^2} + (y + \frac{\varepsilon d}{1 - \varepsilon^2})^2 &= \frac{(1 - \varepsilon^2)d^2 + \varepsilon^2 d^2}{(1 - \varepsilon^2)^2} = (\frac{d}{1 - \varepsilon^2})^2, \text{ and} \\ \frac{x^2}{\frac{d^2}{1 - \varepsilon^2}} + \frac{(y + \frac{\varepsilon d}{1 - \varepsilon^2})^2}{(\frac{d}{1 - \varepsilon^2})^2} &= 1. \end{aligned}$$

Now proceed as in cases (ii) and (iii) of section 3D to show that the graph of this equation is an ellipse if $\varepsilon < 1$ and a hyperbola if $\varepsilon > 1$. In the case of the ellipse $\varepsilon < 1$, so $\sqrt{1 - \varepsilon^2} < 1$, and hence $\sqrt{1 - \varepsilon^2} > 1 - \varepsilon^2$. So $\frac{d}{\sqrt{1 - \varepsilon^2}} < \frac{d}{1 - \varepsilon^2}$. As in (ii) of section 3D, we'll let $a = \frac{d}{1 - \varepsilon^2}$ and $b = \frac{d}{\sqrt{1 - \varepsilon^2}}$. The above equation becomes $\frac{x^2}{b^2} + \frac{(y + a\varepsilon)^2}{a^2} = 1$. Since $a > b$, the semimajor axis of the ellipse is a and its semiminor axis is b . However now, its focal axis is the y -axis and (see the discussion that concludes section 3C) it is obtained by shifting the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ downward so that its upper focal point is at the origin. The case of the hyperbola $\varepsilon > 1$ is similar. Unlike the situation of (iii) of section 3D where the branches of the hyperbola open to the left and right, the branches of the hyperbola open up and down.

Problem 3.17. The graph of the polar function $r = f(\theta) = \cos \theta$ is sketched in Sol 3.4. A look at

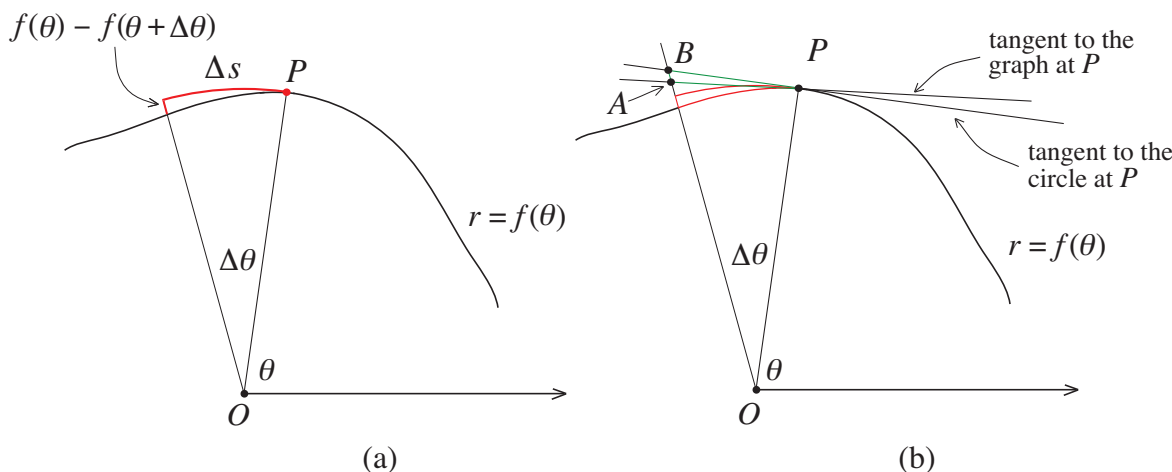


Sol 3.4

the table of Problem 3.6 tells us that over the intervals $0 \leq \theta \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq \theta \leq \pi$, $\pi \leq \theta \leq \frac{3\pi}{2}$, and $\frac{3\pi}{2} \leq \theta \leq 2\pi$, the function $r = f(\theta) = \cos \theta$ respectively, decreases, decreases, increases, and increases. This parallels the fact—see Figure 3.4—that $f'(\theta) = -\sin \theta$ is negative over $0 < \theta < \pi$ and positive over $\pi < \theta < 2\pi$.

Problem 3.19. The graph of the function $r = f(\theta) = \cos \theta$ is the circle with radius $\frac{1}{2}$ and center the point $C = (\frac{1}{2}, 0)$ (in both polar and Cartesian coordinates). Since the radius CP is perpendicular to the tangent at P , the angles $\gamma = \gamma(\theta)$ and $\angle OPC$ add to $\frac{\pi}{2}$. See Sol 3.5b. Since the triangle $\triangle OCP$ is isosceles, $\angle OPC = \theta$. So $\gamma(\theta) + \theta = \frac{\pi}{2}$ and $\gamma(\theta) - \frac{\pi}{2} = -\theta$. To verify $f'(\theta) = f(\theta) \cdot \tan(\gamma(\theta) - \frac{\pi}{2})$, it remains to check that $-\sin \theta = (\cos \theta)(\tan(-\theta))$. But this is so since $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{\sin \theta}{\cos \theta}$.

Problem 3.21. We'll quickly recall the basic situation. Let $P = (f(\theta), \theta)$ be any point on the graph of a polar function $f(\theta)$ with $f(\theta) \neq 0$ (so that P is not the origin O). The point $(f(\theta + \Delta\theta), \theta + \Delta\theta)$ is obtained by adding a small positive angle $\Delta\theta$ to θ . The diagram of Sol 3.6a shows the circular arc (in red) with center O and radius $f(\theta)$ between the segments determined by θ and $\theta + \Delta\theta$. Sol 3.6a sketches a situation (of current interest) where the graph of the function lies below the red circular



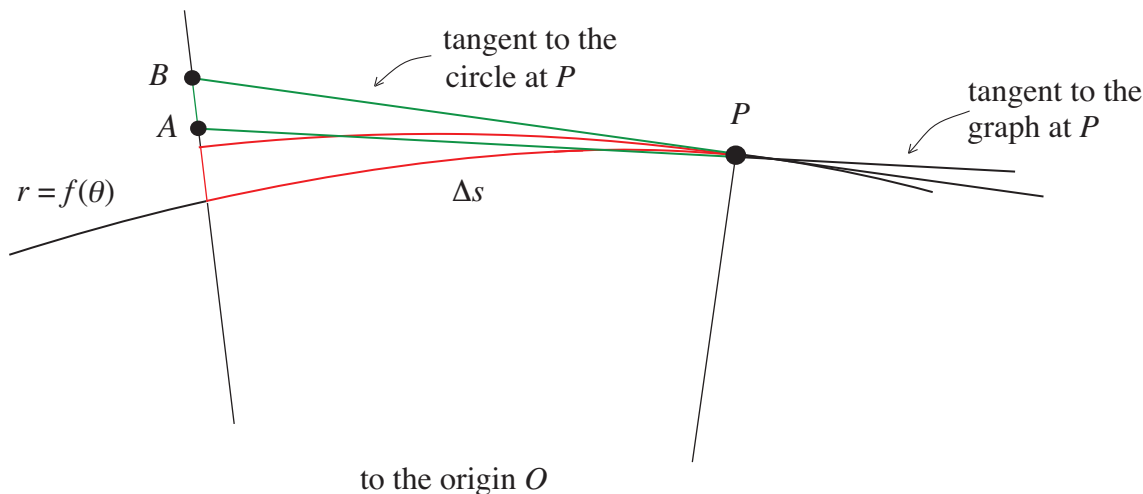
Sol 3.6

arc. As in the situation of Figure 3.22a, the curving triangle of the diagram is the *beak* at P . The radian measure of $\Delta\theta$ is $\Delta\theta = \frac{\Delta s}{f(\theta)}$, where Δs is the length of the circular arc. So $\frac{1}{\Delta\theta} = \frac{1}{\Delta s} \cdot f(\theta)$. After a substitution,

$$\frac{f(\theta + \Delta\theta) - f(\theta)}{\Delta\theta} = \frac{f(\theta + \Delta\theta) - f(\theta)}{\Delta s} \cdot f(\theta).$$

Put in the tangent line to the graph of $r = f(\theta)$ at P , and let A be the point of intersection of the tangent with the ray determined by $\theta + \Delta\theta$. Also put in the tangent line to the circle at P , and let B be the point of intersection of this tangent and the same ray. The two tangent lines and the ray form the triangle $\triangle APB$ that we've called the *triangle* at P . In Sol 3.6b, the beak at P is drawn in red and the triangle at P in green.

We'll now push $\Delta\theta$ to 0 and investigate $\lim_{\Delta\theta \rightarrow 0} \frac{f(\theta + \Delta\theta) - f(\theta)}{\Delta\theta}$. As $\Delta\theta$ is pushed to 0, the segment OAB rotates toward the segment OP . Both the beak at P and the triangle at P shrink



Sol 3.7

in the direction of their tips at P . The shrinking triangle approximates the shrinking beak better and better as the gap between OBA and OP closes. See the diagram of Sol 3.7. In the process, Δs gets closer to BP and $f(\theta) - f(\theta + \Delta\theta) = -(f(\theta + \Delta\theta) - f(\theta))$ to AB . Therefore, as $\Delta\theta$ is pushed to 0,

$$\frac{-(f(\theta + \Delta\theta) - f(\theta))}{\Delta s} \text{ closes in on the ratio } \frac{AB}{BP}.$$

Because the tangent line to a circle at a point is perpendicular to its radius to the point, we know that the angle at P between PO and PB is $\frac{\pi}{2}$. So as $\Delta\theta$ shrinks to 0, the angle $\angle PBA$ approaches $\frac{\pi}{2}$, and the triangle $\triangle APB$ approaches a right triangle with right angle at B . It follows that the ratio $\frac{AB}{BP}$ closes in on the tangent of the angle $\angle APB$. Because $\angle APO = \gamma$ and $\angle BPO = \frac{\pi}{2}$, the angle $\angle APB = \frac{\pi}{2} - \gamma$. By putting it all together, we have demonstrated that as $\Delta\theta$ shrinks to 0

$$\frac{-(f(\theta + \Delta\theta) - f(\theta))}{\Delta s} \text{ closes in on } \frac{AB}{BP} \text{ and this in turn on } \tan(\frac{\pi}{2} - \gamma).$$

Since $\tan(\frac{\pi}{2} - \gamma) = -\tan(\gamma - \frac{\pi}{2})$ we have verified that

$$\lim_{\Delta\theta \rightarrow 0} \frac{f(\theta + \Delta\theta) - f(\theta)}{\Delta s} = \tan(\gamma - \frac{\pi}{2}).$$

We have now arrived at the conclusion $f'(\theta) = f(\theta) \cdot \tan(\gamma(\theta) - \frac{\pi}{2})$ in the case of Sol 3.6a.

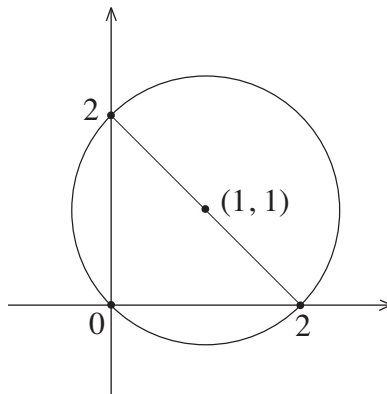
Problem 3.23. Differentiating $f(\theta) = f(0)e^{(\tan(\gamma - \frac{\pi}{2}))\theta}$ with the chain rule gives us

$$f'(\theta) = f(0)e^{\tan(\gamma - \frac{\pi}{2})\theta} \cdot \frac{d}{d\theta}(\tan(\gamma - \frac{\pi}{2})\theta) = f(0)e^{\tan(\gamma - \frac{\pi}{2})\theta} \cdot \tan(\gamma - \frac{\pi}{2}) = f(\theta) \cdot \tan(\gamma - \frac{\pi}{2}).$$

Problem 3.25. We see from the graph of $r = f(\theta) = \sin \theta$ of Figure 3.14 and the area formula of section 3G that the integral $\int_0^\pi \frac{1}{2} \sin^2 \theta d\theta$ is equal to the area of a circle of radius $\frac{1}{2}$. So its value is $\pi(\frac{1}{2})^2 = \frac{\pi}{4}$. Using the equality $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$, we get

$$\int_0^\pi \frac{1}{2} \sin^2 \theta d\theta = \int_0^\pi \frac{1}{4}(1 - \cos 2\theta) d\theta = \frac{1}{4}(\theta - \frac{1}{2} \sin 2\theta) \Big|_0^\pi = \frac{\pi}{4}.$$

Problem 3.27. The circle $(x - 1)^2 + (y - 1)^2 = 2$ has center $(1, 1)$ and radius $\sqrt{2}$. Its graph is sketched in Sol 3.10. The Cartesian points $(2, 0)$ and $(0, 2)$ are on the circle and the segment joining them is on the line $y = -x + 2$. It follows that $(1, 1)$ is on this line as well so that the segment is a



Sol 3.10

diameter of the circle. To find the equivalent polar equation of the circle, we'll let $x = r \cos \theta$ and $y = r \sin \theta$ to get

$$\begin{aligned} (x - 1)^2 + (y - 1)^2 &= (r \cos \theta - 1)^2 + (r \sin \theta - 1)^2 \\ &= (r^2 \cos^2 \theta - 2r \cos \theta + 1) + (r^2 \sin^2 \theta - 2r \sin \theta + 1) \\ &= r^2 - 2r(\cos \theta + \sin \theta) + 2. \end{aligned}$$

So the polar equation of the circle is $r^2 - 2r(\cos \theta + \sin \theta) = 0$. Assuming that $r \neq 0$, we get $r = 2(\cos \theta + \sin \theta)$. For $\theta = \frac{3\pi}{4}$, $r = 2(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) = 0$ so that the origin is on the graph of $r = f(\theta) = 2(\cos \theta + \sin \theta)$. Hence the graph of $r = f(\theta) = 2(\cos \theta + \sin \theta)$ is the entire circle.

A look at Sol 3.10 tells us that $\int_0^{\frac{\pi}{2}} \frac{1}{2} f(\theta)^2 d\theta$ is the area consisting of half the circle plus the area of the right triangle with base and height both equal to 2. So

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} f(\theta)^2 d\theta = \frac{1}{2} \pi (\sqrt{2})^2 + \frac{1}{2} (2 \cdot 2) = \pi + 2.$$

It also follows from the figure that $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$ is equal one-half of the circumference of the circle. So the value of this integral is $\frac{1}{2} (2\pi\sqrt{2}) = \sqrt{2}\pi$.

The two integrals can be solved directly. Since $f(\theta) = 2(\cos \theta + \sin \theta)$,

$$f(\theta)^2 = 4(\cos^2 \theta + 2 \cos \theta \sin \theta + \sin^2 \theta) = 4(1 + 2 \sin \theta \cos \theta).$$

Therefore $\int_0^{\frac{\pi}{2}} \frac{1}{2} f(\theta)^2 d\theta = \int_0^{\frac{\pi}{2}} 2(1 + 2 \cos \theta \sin \theta) d\theta = (2\theta + 2 \sin^2 \theta) \Big|_0^{\frac{\pi}{2}} = \pi + 2$, as before. Since $f'(\theta) = 2(-\sin \theta + \cos \theta)$, we get

$$f'(\theta)^2 = 4(\sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta) = 4(1 - 2 \sin \theta \cos \theta).$$

So $f(\theta)^2 + f'(\theta)^2 = 8$ and $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = 2\sqrt{2}\theta \Big|_0^{\frac{\pi}{2}} = \sqrt{2}\pi$.

Problem 3.29. After writing $r = f(\theta) = \frac{3}{\sin \theta + 2 \cos \theta}$ as $r \sin \theta + 2r \cos \theta = 3$, we see that the graph of this polar function is the line $y = -2x + 3$ with slope -2 and y -intercept 3 sketched in Sol 3.9c. The integral $\int_0^{\frac{\pi}{2}} \frac{1}{2} f(\theta)^2 d\theta$ is the area of the right triangle bounded by the graph of $y = -2x + 3$ and the x - and y -axes. It follows that

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} f(\theta)^2 d\theta = \frac{1}{2} \left(\frac{3}{2} \cdot 3 \right) = \frac{9}{4}.$$

The integral $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$ is the length of the hypotenuse of this triangle. So by the Pythagorean theorem,

$$\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \sqrt{\left(\frac{3}{2}\right)^2 + 3^2} = \sqrt{\frac{45}{4}} = \frac{3}{2}\sqrt{5}.$$

Problem 3.31. Section 3D tells us that the graph of $r = f(\theta) = \frac{4}{1 + \frac{1}{3} \cos \theta}$ is an ellipse with eccentricity $\varepsilon = \frac{1}{3}$ and $d = 4$. By part (ii) of this section the semimajor and semiminor axes of this ellipse are $a = \frac{d}{1-\varepsilon^2} = \frac{4}{1-(\frac{1}{3})^2} = 4 \cdot \frac{9}{8} = \frac{9}{2}$ and $b = \frac{d}{\sqrt{1-\varepsilon^2}} = \frac{4}{\sqrt{1-(\frac{1}{3})^2}} = 4 \cdot \sqrt{\frac{9}{8}} = 4 \cdot \sqrt{\frac{9 \cdot 2}{8 \cdot 2}} = 3\sqrt{2}$.

The area of an ellipse with semimajor and semiminor axes a and b is $ab\pi$. Since $\int_0^\pi \frac{1}{2} \left(\frac{4}{1 + \frac{1}{3} \cos \theta} \right)^2 d\theta$ is one-half the area of the ellipse being considered, it follows that

$$\int_0^\pi \frac{8}{(1 + \frac{1}{3} \cos \theta)^2} d\theta = \int_0^\pi \frac{1}{2} \left(\frac{4}{1 + \frac{1}{3} \cos \theta} \right)^2 d\theta = \frac{1}{2} \left(\frac{9}{2} \right) (3\sqrt{2}) \pi = \frac{27}{4} \sqrt{2} \pi.$$

Problem 3.33. For the equiangular spiral $f(\theta) = \frac{1}{7} e^{\frac{\theta}{\sqrt{3}}}$, $f'(\theta) = \frac{1}{7} e^{\frac{\theta}{\sqrt{3}}} \cdot \frac{1}{\sqrt{3}}$, so that by the discussion of equiangular spirals in section 3H, $\tan(\gamma(\theta) - \frac{\pi}{2}) = \frac{f'(\theta)}{f(\theta)} = \frac{1}{\sqrt{3}}$. Since $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$, it follows that $\gamma(\theta) - \frac{\pi}{2} = \frac{\pi}{6}$ and hence that $\gamma(\theta) = \frac{2\pi}{3}$. The graph of the spiral in Sol 3.11 below was sketched by the polar graphing calculator

<https://www.desmos.com/calculator/ms3eghkkgz>

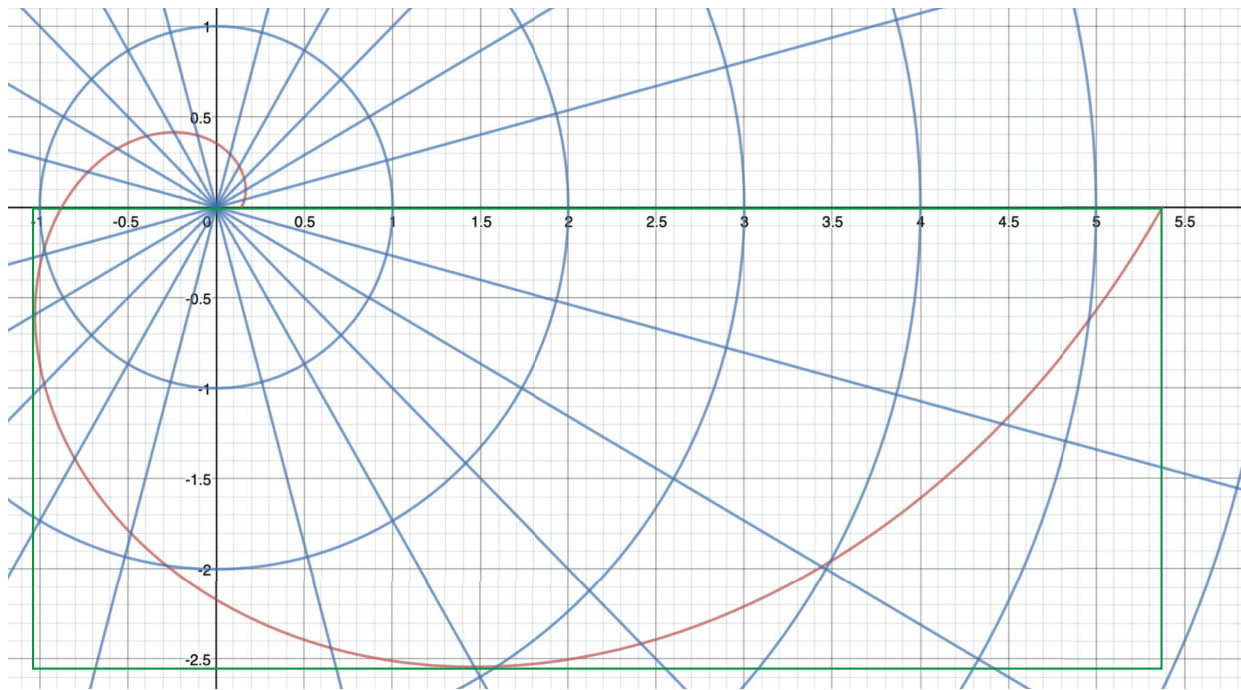
Since $f'(\theta) = \frac{1}{7\sqrt{3}} e^{\frac{\theta}{\sqrt{3}}}$, we see that the length of the spiral from $\theta = 0$ to $\theta = 2\pi$ is given by

$$\begin{aligned} \int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta &= \int_0^{2\pi} \sqrt{\frac{1}{7^2} e^{\frac{2\theta}{\sqrt{3}}} + \frac{1}{3 \cdot 7^2} e^{\frac{2\theta}{\sqrt{3}}}} d\theta = \frac{1}{7} \int_0^{2\pi} \sqrt{\frac{4}{3} e^{\frac{2\theta}{\sqrt{3}}}} d\theta \\ &= \frac{2}{7\sqrt{3}} \int_0^{2\pi} e^{\frac{1}{\sqrt{3}} \theta} d\theta = \frac{2}{7\sqrt{3}} \left(\sqrt{3} e^{\frac{1}{\sqrt{3}} \theta} \Big|_0^{2\pi} \right) = \frac{2}{7} (e^{\frac{2\pi}{\sqrt{3}}} - 1) \approx 10.46. \end{aligned}$$

The area that the spiral (along with the polar axis) encloses is

$$\int_0^{2\pi} \frac{1}{2} f(\theta)^2 d\theta = \int_0^{2\pi} \frac{1}{2} \left(\frac{1}{7^2} e^{\frac{2\theta}{\sqrt{3}}} \right) d\theta = \frac{1}{2 \cdot 7^2} \int_0^{2\pi} e^{\frac{2\theta}{\sqrt{3}}} d\theta = \frac{1}{2 \cdot 7^2} \left(\frac{\sqrt{3}}{2} e^{\frac{2\theta}{\sqrt{3}}} \right) \Big|_0^{2\pi} = \frac{\sqrt{3}}{2^2 \cdot 7^2} (e^{\frac{4\pi}{\sqrt{3}}} - 1) \approx 12.50.$$

The fact that the two shorter sides plus the bottom side of the green rectangle of the figure determined by the intervals $[-1.05, 5.4]$ on the x -axis and $[0, -2.55]$ on the y -axis (see the figure)



Sol 3.11. The graph of $r = \frac{1}{7}e^{\frac{\theta}{\sqrt{3}}}$

add to $2.55 + 2.55 + 6.45 = 11.55$ and that its area is $2.55 \cdot 6.45 \approx 16.45$ confirms the reasonableness of the two answers.

Problem 3.35. What about the argument that led to the conclusion that the length L of the polar graph of $r = f(\theta)$ between $\theta = a$ and $\theta = b$ is equal to

$$L = \int_a^b |f(\theta)| d\theta?$$

This formula agrees with the correct version of section 3F in general only if $f'(\theta) = 0$, so only if $f(\theta) = c$ is a constant, and hence the graph of $r = f(\theta)$ is a circle with center the origin O .

Let's consider the equiangular spiral $r = f(\theta) = \frac{1}{4}e^{\tan(\gamma - \frac{\pi}{2})\theta} = \frac{1}{4}e^{(\tan \frac{\pi}{4})\theta} = \frac{1}{4}e^\theta$. See Sol 3.12a. Because $f'(\theta) = \frac{1}{4}e^\theta$,

$$\int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \int_0^{2\pi} \sqrt{2f(\theta)^2} d\theta = \sqrt{2} \int_0^{2\pi} |f(\theta)| d\theta.$$

So the correct value differs from the incorrect value by a factor of $\sqrt{2}$.

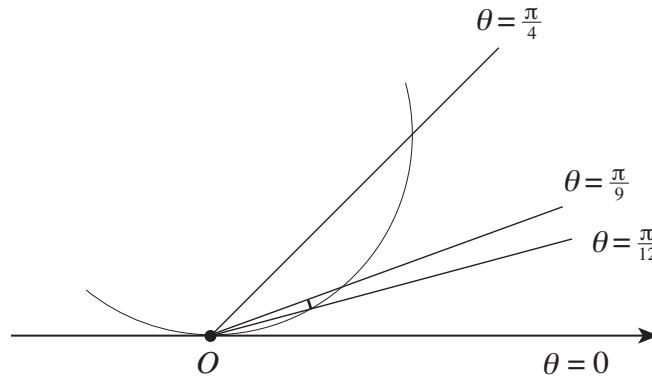
With regard to Figure 3.25 the simple fact is that for some graphs (or parts of graphs) the length $f(\theta_i) d\theta$ of the circular arc does not approximate the length of the graph between the rays determined by θ_i and θ_{i+1} with sufficient accuracy. To give a quantitative sense of this, we'll experiment with $f(\theta) = \sin \theta$ rather than $f(\theta) = \frac{1}{4}e^\theta$. The graph of $f(\theta) = \sin \theta$ is the circle sketched in Figure 3.14. Notice that as θ moves from 0 through small positive angles, the point (r, θ) on the graph recedes quickly from the origin. This part of the graph illustrates the problem of "sufficient accuracy." Consider the rays $\theta = \frac{\pi}{12}$ and $\theta = \frac{\pi}{9}$. The ray $\theta = \frac{\pi}{12}$ cuts the graph at the point $(\sin \frac{\pi}{12}, \frac{\pi}{12}) \approx (0.26, 0.26)$. By the length formula for a circular arc, the circular arc centered at O from this point to the ray $\theta = \frac{\pi}{9}$ has length

$$f\left(\frac{\pi}{12}\right)\left(\frac{\pi}{9} - \frac{\pi}{12}\right) = \left(\sin \frac{\pi}{12}\right)\left(\frac{\pi}{9} - \frac{\pi}{12}\right) = \left(\sin \frac{\pi}{12}\right)\left(\frac{\pi}{36}\right) \approx 0.0226.$$

Let's compare this against the length of the graph of the function between these two rays. This length is

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \sqrt{\sin^2 \theta + \cos^2 \theta} d\theta = \theta \Big|_{\frac{\pi}{12}}^{\frac{\pi}{9}} = \left(\frac{\pi}{9} - \frac{\pi}{12}\right) = \frac{\pi}{36} \approx 0.0873,$$

or about 4 times the length of the circular arc. The diagram in Sol 3.13 illustrates the difference

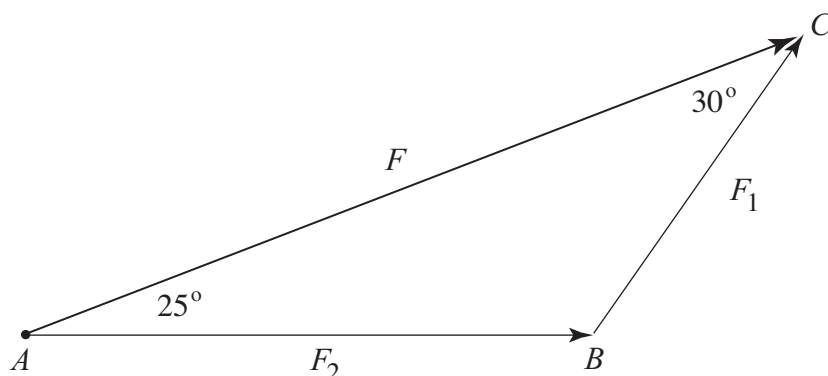


Sol 3.13

between the lengths of this part of the graph of $f(\theta) = \sin \theta$ and that of the corresponding circular arc.

Chapter 4. Solutions of Odd Problems

Problem 4.1. The figure below is derived from Figure 4.23. The angle at B is $180^\circ - 25^\circ - 30^\circ = 125^\circ$. With F_1 and F_2 the magnitudes of the two forces and $F = 115$ pounds the magnitude of their



Sol 4.1

resultant, we get by applying the law of sines to the triangle ABC that

$$\frac{\sin 25^\circ}{F_1} = \frac{\sin 30^\circ}{F_2} = \frac{\sin 125^\circ}{115}.$$

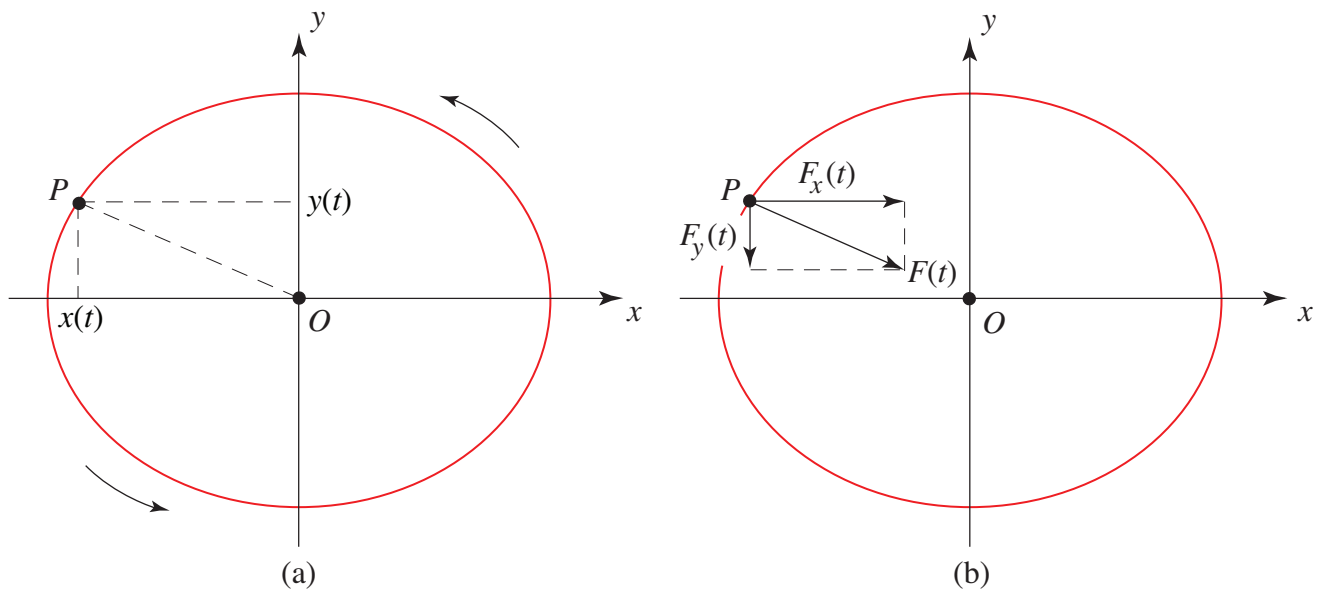
So $F_1 = \sin 25^\circ \frac{115}{\sin 125^\circ} \approx (\frac{0.42262}{0.81915})115 \approx 59.33$ pounds and $F_2 = \sin 30^\circ \frac{115}{\sin 125^\circ} \approx (\frac{0.5}{0.81915})115 \approx 70.19$ pounds.

Problem 4.3. Focus on Figure 4.4a. Given that the string has a length of $PH = 1.6$ meters and that $\theta = 60^\circ$, the radius PS of the circular path of the object P satisfies $\frac{PS}{PH} = \cos 60^\circ = 0.5$, so that $PS = 0.8$. With the mass of P equal to 2 kg, its weight is $W = 2g$, where g is the gravitational constant in MKS. Taking $g = 9.8$ m/sec², we get $W = 2 \cdot 9.8 = 19.6$ newtons. This is the magnitude of the vertical component of F_H , so that $F_H \sin 60^\circ = 19.6$. Since $\sin 60^\circ = \frac{\sqrt{3}}{2}$, we get $F_H = \frac{19.6}{\sin 60^\circ} = 19.6 \frac{2}{\sqrt{3}} \approx 22.62$ newtons. The magnitude of the centripetal force on P is $F = F_H \cos 60^\circ \approx 11.36$ newtons.

Problem 4.5. If the angle $\theta(t)$ of Figure 4.13 increases at a constant rate, then by the formula $r(t)^2 \theta'(t) = 2\kappa$ of section 4D, $r(t)^2$ is constant and hence $r(t)$ is constant. This means that the orbit is a circle. A look at the formula $F(t) = m[\frac{4\kappa^2}{r(t)^3} - \frac{d^2 r}{dt^2}] = \frac{4m\kappa^2}{r(t)^3}$ of section 4D informs us that $F(t)$ is constant as well.

Problem 4.7. The equation that defines the polar function $r = f(\theta) = \frac{d}{a \sin \theta + b \cos \theta}$ can be written as $a(r \sin \theta) + b(r \cos \theta) = d$, so that its Cartesian version is the equation of the line $ay + bx = d$. Since $d \neq 0$, the origin is not on the line. Note that $g(\theta) = \frac{1}{f(\theta)} = \frac{1}{d}(a \sin \theta + b \cos \theta)$. So $g'(\theta) = \frac{1}{d}(a \cos \theta - b \sin \theta)$ and $g''(\theta) = \frac{1}{d}(-a \sin \theta - b \cos \theta)$. It follows that $g''(\theta) = -g(\theta)$. An application of the second form of the centripetal force equation of section 4D tells us that $F(t) = 0$.

Problem 4.9. The position of the point-mass P of mass m moving in an xy -plane is given by the equations $x(t) = a \cos \omega t$ and $y(t) = b \sin \omega t$, where $t \geq 0$ is time, and a , b , and ω are constants



Sol 4.6

with $a \geq b > 0$, and $\omega > 0$. That the distance between P and O at any time t is $r(t) = \sqrt{a^2 \cos^2 \omega t + b^2 \sin^2 \omega t}$ follows directly from the Pythagorean theorem. A substitution shows quickly that the x - and y -coordinates of P satisfy the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so that P moves on this ellipse with semimajor axis a and semiminor axis b . Studying the x - and y -coordinates of P for increasing $t \geq 0$ in combination with Figures 3.4 and 3.5, tells us that the point starts at $(a, 0)$ and that it moves counterclockwise around the ellipse.

By referring to Sol 4.6 and following what was done in Example 4.7, we can conclude that at any point $(x(t), y(t))$ the slopes of the segment PO and the vector determined by the resultant of the forces $F_x(t) = mx''(t)$ and $F_y(t) = my''(t)$ are the same. It follows that this resultant force on P is a centripetal force in the direction of the origin O and that it has magnitude $F(t) = \sqrt{F_x(t)^2 + F_y(t)^2} = m\omega^2 \sqrt{a^2 \cos^2 \omega t + b^2 \sin^2 \omega t} = m\omega^2 r(t)$.

The fact that P starts at time $t = 0$ and completes its first revolution when t satisfies $\omega t = 2\pi$, tells us that the period of the orbit is $T = \frac{2\pi}{\omega}$. Since the area of the ellipse is $A = ab\pi$, it follows that Kepler constant of the orbit is $\kappa = \frac{A}{T} = (ab\pi)(\frac{\omega}{2\pi}) = \frac{ab\omega}{2}$. Since the force on P is centripetal, $F(t)$ and $r(t)$ satisfy the force equation $F(t) = m[\frac{4\kappa^2}{r(t)^3} - \frac{d^2 r}{dt^2}]$ of section 4D. (This can also be verified directly for the given $r(t)$ with a labor-intensive calculus exercise.) The case of a circular orbit was dealt with in Example 4.7, so we'll now suppose that $a > b$. Assume, if possible, that $F(t)$ satisfies an inverse square law $F(t) = C\frac{m}{r(t)^2}$. Then $C\frac{m}{r(t)^2} = m\omega^2 r(t)$ and hence $r(t)^3 = \frac{C}{\omega^2}$. The consequence that $r(t)$ is constant contradicts the fact that the orbit is not a circle. So $F(t)$ does not satisfy an inverse square law. Conclusion B of section 4G leaves us with the conclusion that the center of the centripetal force cannot be a focal point of the ellipse. But this is consistent with what we already know, namely that it is the center of the ellipse that is the center of the centripetal force of this example. In the circular case, the center of the ellipse is its one and only focal point and as we saw in Example 4.10, $F(t)$ does satisfy an inverse square law.

The next few problems turn to the motion of a thrown object near Earth's surface. The solutions of the problems require formulas developed in the paragraph *Projectile Motion on Earth*.

Problem 4.11. We're assuming that the terrain is flat and horizontal and the x -axis lies along the ground, so that $y(t) = 0$ at the time t of impact. The expression $y(t) = -\frac{g}{2}t^2 + (v_0 \sin \varphi)t + y_0$ and the quadratic formula tell us that the time t at which impact occurs is

$$t = \frac{-(v_0 \sin \varphi) \pm \sqrt{v_0^2 \sin^2 \varphi - 4(-\frac{g}{2})y_0}}{-g} = \frac{(v_0 \sin \varphi) \pm \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}}{g}.$$

Since $t \geq 0$, the $+$ option applies and the time of impact is $t_{\text{imp}} = \frac{(v_0 \sin \varphi) + \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}}{g}$. Since $x(t) = (v_0 \cos \varphi)t$, the projectile impacts

$$x(t_{\text{imp}}) = \frac{v_0}{g} \cos \varphi [v_0 \sin \varphi + \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}].$$

downrange. The distance $R = \frac{v_0}{g} \cos \varphi [v_0 \sin \varphi + \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}]$ is the range of the projectile.

Assume that $y_0 = 0$. The trig formula $\sin 2\varphi = 2 \sin \varphi \cos \varphi$ tells us that

$$R = \frac{v_0}{g} \cos \varphi [v_0 \sin \varphi + \sqrt{v_0^2 \sin^2 \varphi}] = \frac{v_0}{g} \cos \varphi (2v_0 \sin \varphi) = \frac{v_0^2}{g} \sin 2\varphi.$$

So when $y_0 = 0$, the maximal range is achieved for $\varphi = \frac{\pi}{4}$ and is equal to $R_{\text{max}} = \frac{v_0^2}{g}$.

Problem 4.13. The speed of the projectile at any time t is

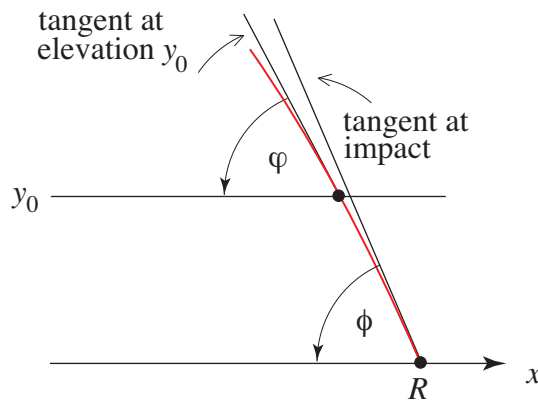
$$v(t) = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(v_0 \cos \varphi)^2 + (-gt + v_0 \sin \varphi)^2} = \sqrt{v_0^2 + g^2 t^2 - 2g(v_0 \sin \varphi)t}.$$

The slope of the trajectory at any time t is $\frac{y'(t)}{x'(t)} = \frac{-gt + v_0 \sin \varphi}{v_0 \cos \varphi}$. If the terrain is flat and horizontal, then by Problem 4.11, the time of impact is $t = \frac{(v_0 \sin \varphi) + \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}}{g}$. With t the time of impact,

$$\begin{aligned} v(t) &= \sqrt{v_0^2 + [(v_0 \sin \varphi) + \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}]^2 - 2(v_0 \sin \varphi)[(v_0 \sin \varphi) + \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}]} \\ &= \sqrt{v_0^2 + 2(v_0 \sin \varphi)\sqrt{v_0^2 \sin^2 \varphi + 2gy_0} + (v_0^2 \sin^2 \varphi + 2gy_0) - v_0^2 \sin^2 \varphi - 2(v_0 \sin \varphi)\sqrt{v_0^2 \sin^2 \varphi + 2gy_0}} \\ &= \sqrt{v_0^2 + 2gy_0}, \text{ and} \end{aligned}$$

$$\frac{y'(t)}{x'(t)} = \frac{-(v_0 \sin \varphi) - \sqrt{v_0^2 \sin^2 \varphi + 2gy_0} + v_0 \sin \varphi}{v_0 \cos \varphi} = \frac{-\sqrt{v_0^2 \sin^2 \varphi + 2gy_0}}{v_0 \cos \varphi} = -\sqrt{\frac{v_0^2 \sin^2 \varphi}{v_0^2 \cos^2 \varphi} + \frac{2gy_0}{v_0^2 \cos^2 \varphi}} = -\sqrt{\tan^2 \varphi + \frac{2gy_0}{v_0^2 \cos^2 \varphi}}.$$

The formula $\frac{y'(t)}{x'(t)} = -\sqrt{\tan^2 \varphi + \frac{2gy_0}{v_0^2 \cos^2 \varphi}}$ confirms what Figure 4.27 illustrates and this is that the slope of the trajectory at the point of impact is negative. Refer to the solution of Problem 4.10 and the fact that the projectile reaches its maximum height when $t = \frac{v_0}{g} \sin \varphi$. The x -coordinate of the position of the projectile at this time is $(v_0 \cos \varphi)(\frac{v_0}{g} \sin \varphi) = \frac{v_0^2}{g} \sin \varphi \cos \varphi = \frac{v_0^2}{2g} \sin 2\varphi$. A general property of the parabola tells us and Figure 4.27 illustrates that the trajectory of the projectile is symmetric about the vertical line $x = \frac{v_0^2}{2g} \sin 2\varphi$. It follows that the angle of the tangent to the path of the projectile at the elevation of y_0 of its descent is $-\varphi$. Let ϕ be the angle of the path of the projectile at impact. The fact that $-\sqrt{\tan^2 \varphi + \frac{2gy_0}{v_0^2 \cos^2 \varphi}} \leq -\sqrt{\tan^2 \varphi} = -\tan \varphi$ means that the angle at impact is greater (in the sense of absolute value) than φ . Figure Sol 4.7 illustrates what



Sol 4.7

has been observed. It is derived from Figure 4.27.

We take up issues that remain from the discussion in the paragraph *Tossing the Hammer*. Putting the data provided in the paragraph into the formula $R = \frac{v_0}{g} \cos \varphi [v_0 \sin \varphi + \sqrt{v_0^2 \sin^2 \varphi + 2gy_0}]$ for the range of a projectile, we get

$$R = \frac{30.7}{9.81} (\cos 39.9^\circ) [30.7 \sin 39.9^\circ + \sqrt{(30.7^2)(\sin^2 39.9^\circ) + 2(9.81)(1.66)}] \approx 96.50 \text{ meters}$$

for the distance that Yuriy Sedykh's record setting throw would have attained in an air resistance free environment. (The discrepancy between the 96.50 meters and the 96.56 meters of the text is due to the differing round-off strategies that were used.) This tells us that air resistance and the retarding effect of the wire and the handle reduced this theoretical value of the record throw by $96.50 - 86.74 = 9.76$ meters.

The force with which the hammer pulled on Yuriy just before the moment of the hammer's release was computed in the text to have been 3767 newtons. As was observed, this force was much greater than Yuriy's weight of about 1079 newtons. So why didn't Yuriy fly off with the ball just before he released it? This question has the same answer as the question as to why the Earth and Moon are not pushed to a collision by the gravitational force that attracts them toward each other. The answer is that the force of attraction between them is counterbalanced by the rapid motion of both around their common barycenter.

We turn next to the paragraph *Supersized Reflecting Telescopes* and the solution of Problem 4.15. Recall the fact that the parabolic boundary of the mirror of the GMT is given by the function $y = f(x) = \frac{2\pi^2\tau^2}{g}x^2$. A look at the study of the parabola in Chapter 1C tells us that the focal point of the parabola with equation $x^2 = 4cy$, or equivalently $y = \frac{1}{4c}x^2$, where $c > 0$ is a constant, is the point $(0, c)$. Solving $\frac{1}{4c} = \frac{2\pi^2\tau^2}{g}$ for c gives $c = \frac{g}{8\pi^2\tau^2}$, and tells us that the focal point of the parabola of the mirror is a distance $\frac{g}{8\pi^2\tau^2}$ meters above its lowest point.

Problem 4.15. The data $g = 9.80 \text{ m/sec}^2$ and $\tau = \frac{1}{12}$ revolutions per second, tells us that $\frac{2\pi^2\tau^2}{g} \approx 0.01399 \approx 0.014$, and provides the approximate formula $f(x) = 0.014x^2$ for the mirror's parabola. That the vertical distance between the horizontal plane at the mirror's rim and its central hole is $f(4.2) - f(1.15) \approx 0.014[(4.2)^2 - (1.15)^2] \approx 0.23 \text{ m}$ follows directly from Figure 4.32. The

focal point of the central mirror is the point $(0, \frac{g}{8\pi^2\tau^2}) \approx (0, \frac{(9.80)(12^2)}{8(9.87)}) \approx (0, 17.87)$. So the focal point of the mirror lies about 18 meters above its central hole.

The next problem gives an example of a mass of uniform density that attracts a point-mass with a gravitational force of a magnitude that is distinctly different from the value provided by Newton's law of universal gravitation.

Problem 4.17. The uniform density of the cylinder—mass over volume—is $\rho = \frac{M}{\pi R^2 h}$. Let's focus on one of the many typical thin discs of thickness dx that the cylinder has been sliced into. See Figure 4.34. The volume of the disc is $\pi R^2 dx$, so that its mass is $\pi R^2 dx \cdot \rho$. The point-mass of mass m lies on the central axis of the cylinder at a distance $c - x$ from the disc's center. By the conclusion of Problem 4.16, the gravitational force with which the disc attracts the point-mass acts in the direction of the center of the disc with a magnitude of

$$G \frac{2m(\pi R^2 dx \cdot \rho)}{R^2} \left(1 - \frac{c-x}{\sqrt{R^2+(c-x)^2}}\right) = G \cdot 2m\pi \left(\frac{M}{\pi R^2 h}\right) dx \left(1 - \frac{c-x}{(R^2+(c-x)^2)^{\frac{1}{2}}}\right) = G \frac{2mM}{R^2 h} \left(1 - \frac{c-x}{(R^2+(c-x)^2)^{\frac{1}{2}}}\right) dx.$$

Summing things up over all the discs from $x = 0$ to $x = h$, we find that the force with which the cylinder attracts the point mass is directed along the central axis of the cylinder with a magnitude of

$$F = \int_0^h G \frac{2mM}{R^2 h} \left(1 - \frac{c-x}{(R^2+(c-x)^2)^{\frac{1}{2}}}\right) dx = G \frac{2mM}{R^2 h} \int_0^h \left(1 + \frac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}}\right) dx.$$

Since $\int_0^h \left(1 + \frac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}}\right) dx = h + \int_0^h \frac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}} dx$, it remains to compute $\int_0^h \frac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}} dx$.

Start with the substitution $u = x - c$ and $du = dx$, to get $\int_0^h \frac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}} dx = \int_{-c}^{h-c} \frac{u}{(R^2+u^2)^{\frac{1}{2}}} du$.

Then let $v = R^2 + u^2$ and $dv = 2u du$ so that

$$\int_{-c}^{h-c} \frac{u}{(R^2+u^2)^{\frac{1}{2}}} du = \int_{R^2+c^2}^{R^2+(h-c)^2} \frac{\frac{1}{2}v^{-\frac{1}{2}} dv}{2} = v^{\frac{1}{2}} \Big|_{R^2+c^2}^{R^2+(h-c)^2} = (R^2 + (h-c)^2)^{\frac{1}{2}} - (R^2 + c^2)^{\frac{1}{2}}.$$

After putting things together, we have determined that the magnitude of the force of attraction of the cylinder of Figure 4.34 on the point-mass is

$$F = G \frac{2mM}{R^2 h} \left[h - (R^2 + c^2)^{\frac{1}{2}} + (R^2 + (c-h)^2)^{\frac{1}{2}} \right].$$

The center of mass of the cylinder is the point $\frac{1}{2}h$ on the horizontal axis, so that the value that Newton's formula provides for the magnitude of this force is $F = G \frac{mM}{(c-\frac{1}{2}h)^2}$. When R and h are both very small, then the cylinder can be regarded as a point-mass and the two values are in close agreement. To verify this, use L'Hospital's rule to show that $\lim_{h \rightarrow 0} \frac{h - (R^2 + c^2)^{\frac{1}{2}} + (R^2 + (c-h)^2)^{\frac{1}{2}}}{h} = 1 - \frac{c-h}{(R^2 + (c-h)^2)^{\frac{1}{2}}}$ and combine this result with the conclusion of Problem 4.16.

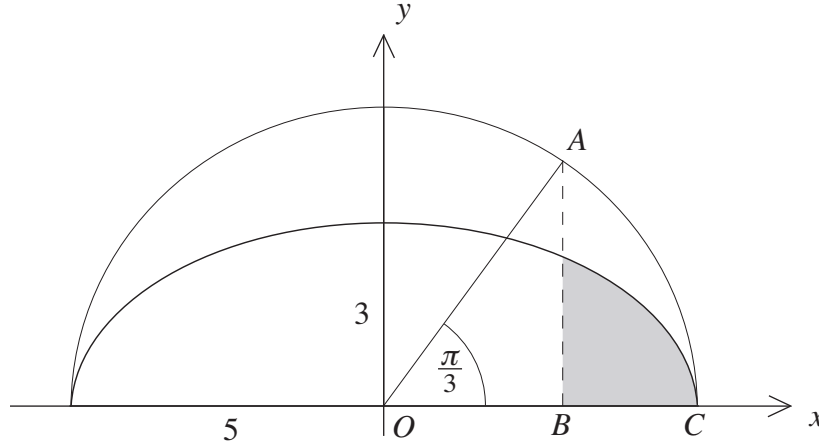
Problem 4.19. Using the estimate 7.35×10^{22} kg for the Moon's mass and a conclusion of Problem 4.18, we get

$$\frac{7.35 \times 10^{22}}{2.1968 \times 10^{19}} \approx 3.35 \times 10^3 \text{ kg/m}^3$$

for the average density of the Moon. It was established in section 4H that Earth's average density is $5.51 \times 10^3 \text{ kg/m}^3$.

Chapter 5. Solutions of Odd Problems

Problem 5.1. Place Figure 5.16 on an xy -coordinate system as shown in Sol 5.1. The angle $\frac{\pi}{3}$ determines the circular sector OAC of radius 5. The area of this circular sector is $\frac{1}{2}(\frac{\pi}{3} \cdot 5^2) = \frac{25\pi}{6}$. The x -coordinate of the point B satisfies $\cos \frac{\pi}{3} = \frac{x}{5}$, so that $x = 5 \cos \frac{\pi}{3} = \frac{5}{2}$. Because $x^2 + y^2 = 5^2$ is an equation of the circle and $f(x) = \sqrt{5^2 - x^2}$ is a function that has the upper part of the circle



Sol 5.1

as its graph, $AB = f(\frac{5}{2}) = \sqrt{5^2 - (\frac{5}{2})^2} = \sqrt{\frac{100-25}{4}} = \frac{5\sqrt{3}}{2}$. Therefore the area under the circle and over the segment BC is equal to $\frac{25\pi}{6} - \frac{1}{2}(\frac{5}{2} \cdot \frac{5\sqrt{3}}{2}) = 25(\frac{\pi}{6} - \frac{\sqrt{3}}{8})$. It follows that

$$\int_{\frac{5}{2}}^5 \sqrt{5^2 - x^2} dx = 25(\frac{\pi}{6} - \frac{\sqrt{3}}{8}).$$

Since $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ is an equation of the ellipse, the function $g(x) = \frac{3}{5}\sqrt{5^2 - x^2}$ has the upper part of the ellipse as its graph. It follows that the shaded area of the figure is equal to

$$\int_{\frac{5}{2}}^5 \frac{3}{5} \sqrt{5^2 - x^2} dx = \frac{3}{5} \int_{\frac{5}{2}}^5 \sqrt{5^2 - x^2} dx = \frac{3}{5} (25(\frac{\pi}{6} - \frac{\sqrt{3}}{8})) = 15(\frac{\pi}{6} - \frac{\sqrt{3}}{8}) \approx 4.61.$$

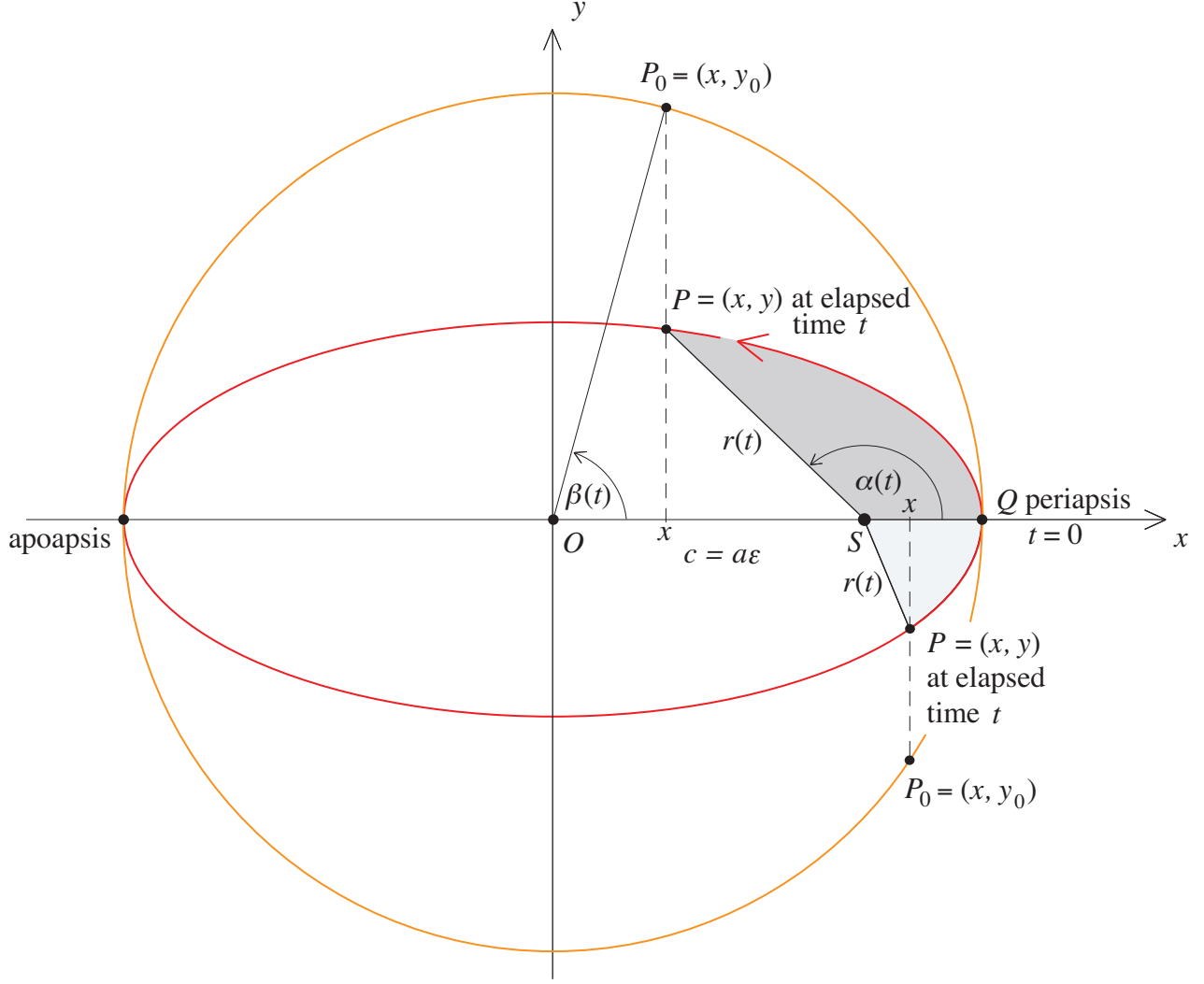
Problem 5.3. Sol 5.2 is a version of Figure 5.3 with P in the first or fourth quadrants, respectively. It includes the surrounding circle with the angle $\beta(t)$ and the point P_0 . The coordinate $x = x(t)$ is now positive. As in the cases dealt with in section 5B, the distance OS between the center and the focus of the ellipse is equal to $c = a\varepsilon$ and $a^2 = b^2 + c^2$. Consider the case where P is in the first quadrant. The Pythagorean theorem tells us that

$$r(t)^2 = (a\varepsilon - x(t))^2 + y(t)^2 = (a\varepsilon)^2 - 2a\varepsilon x(t) + x(t)^2 + y(t)^2 = (a\varepsilon)^2 - 2a\varepsilon x(t) + x(t)^2 + \frac{b^2}{a^2} y_0(t)^2.$$

Since the point P_0 is on the circle of radius a , $x(t)^2 + y_0(t)^2 = a^2$, so that

$$\begin{aligned} r(t)^2 &= (a\varepsilon)^2 - 2a\varepsilon x(t) + x(t)^2 + \frac{b^2}{a^2} (a^2 - x(t)^2) = (a\varepsilon)^2 + b^2 - 2a\varepsilon x(t) + x(t)^2 - \frac{b^2}{a^2} x(t)^2 \\ &= a^2 - 2a\varepsilon x(t) + x(t)^2 - \frac{a^2 - (a\varepsilon)^2}{a^2} x(t)^2 = a^2 - 2a\varepsilon x(t) + x(t)^2 - x(t)^2 + \frac{(a\varepsilon)^2}{a^2} x(t)^2 \\ &= a^2 - 2a\varepsilon x(t) + \varepsilon^2 x(t)^2 = (a - \varepsilon x(t))^2. \end{aligned}$$

Because $a \geq x(t) \geq \varepsilon x(t)$, $a - \varepsilon x(t) \geq 0$. Since $r(t) \geq 0$ and $r(t)^2 = (a - \varepsilon x(t))^2$, it follows that



Sol 5.2

$r(t) = a - \varepsilon x(t)$. The substitution $x(t) = a \cos \beta(t)$ provides the equality $r(t) = a(1 - \varepsilon \cos \beta(t))$. In the case where P is in the first quadrant, the verification of Gauss's equation $\tan \frac{\alpha(t)}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan \frac{\beta(t)}{2}$ is the same as before.

Let's turn to the situation in Sol 5.2 with P in the fourth quadrant. The Pythagorean theorem tells us in this case that

$$r(t)^2 = (x(t)^2 - a\varepsilon)^2 + (-y(t))^2 = (x(t)^2 - a\varepsilon)^2 + \frac{b^2}{a^2}(-y_0(t))^2.$$

Since P_0 is on the circle of radius a , $x(t)^2 + (-y_0(t))^2 = a^2$, and hence

$$\begin{aligned} r(t)^2 &= (x(t)^2 - a\varepsilon)^2 + \frac{b^2}{a^2}(a^2 - (x(t)^2)) = (x(t)^2 - a\varepsilon)^2 + b^2 - \frac{b^2}{a^2}x(t)^2 \\ &= x(t)^2 - 2a\varepsilon x(t) + a^2\varepsilon^2 + b^2 - \frac{b^2}{a^2}x(t)^2 = \frac{a^2-b^2}{a^2}x(t)^2 - 2a\varepsilon x(t) + a^2 \\ &= \varepsilon^2 x(t)^2 - 2a\varepsilon x(t) + a^2 = (a - \varepsilon x(t))^2. \end{aligned}$$

This implies that $r(t) = a - \varepsilon x(t)$, as before. And as before, the substitution $x(t) = a \cos \beta(t)$ provides the equality $r(t) = a(1 - \varepsilon \cos \beta(t))$. In the case where P is in the fourth quadrant, the verification of Gauss's equation $\tan \frac{\alpha(t)}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan \frac{\beta(t)}{2}$ needs to be slightly modified. As before, it relies on the link between $\alpha(t)$ and $\beta(t)$ given by

$$\cos \alpha(t) = \cos(-\alpha(t)) = \cos(2\pi - \alpha(t)) = \frac{x(t) - a\varepsilon}{r(t)} = \frac{a \cos \beta(t) - a\varepsilon}{a(1 - \varepsilon \cos \beta(t))} = \frac{\cos \beta(t) - \varepsilon}{1 - \varepsilon \cos \beta(t)}.$$

Problem 5.5. From $\cos \alpha(t) = \frac{\cos \beta(t) - \varepsilon}{1 - \varepsilon \cos \beta(t)}$ we get

$$\cos \beta(t) - \varepsilon = \cos \alpha(t)(1 - \varepsilon \cos \beta(t)) = \cos \alpha(t) - \cos \alpha(t)(\varepsilon \cos \beta(t)).$$

So $\cos \beta(t) + \varepsilon \cos \alpha(t) \cos \beta(t) = \varepsilon + \cos \alpha(t)$, and hence $\cos \beta(t) = \frac{\varepsilon + \cos \alpha(t)}{1 + \varepsilon \cos \alpha(t)}$. Insert this into the equation $r(t) = a(1 - \varepsilon \cos \beta(t))$ to get

$$r(t) = a(1 - \varepsilon \cos \beta(t)) = a\left(1 - \varepsilon \frac{\varepsilon + \cos \alpha(t)}{1 + \varepsilon \cos \alpha(t)}\right) = a\left(\frac{1 + \varepsilon \cos \alpha(t) - \varepsilon^2 - \varepsilon \cos \alpha(t)}{1 + \varepsilon \cos \alpha(t)}\right) = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \alpha(t)}.$$

By ignoring the fact that α is a function of t , we see that r is the function $r(\alpha) = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \alpha}$ of α .

Problem 5.7. A look at Figure 5.4 tells us that the time t it takes the point-mass P to go from periapsis to apoapsis satisfies $\beta(t) = \pi$. It follows from Kepler's equation $\beta(t) - \varepsilon \sin \beta(t) = \frac{2\pi t}{T}$, that $\pi = \frac{2\pi t}{T}$ and that $t = \frac{T}{2}$. Since T is the period of the orbit, it takes the point-mass P another $\frac{T}{2}$ to return from apoapsis to periapsis.

Problem 5.9. From Figure 5.1 we'll take $T = 0.6152$ years, or $(0.6152)(365.25) = 224.7018$ days, for the period of Venus's orbit and $\varepsilon = 0.0068$ for its eccentricity. Let t_1, t_2 , and t_3 be the number of days it takes for the angle α of Venus's orbit to rotate from 0° to 60° , from 0° to 120° and 0° to 180° , respectively. So $\alpha(t_1) = \frac{\pi}{3}$, $\alpha(t_2) = \frac{2\pi}{3}$, and $\alpha(t_3) = \pi$. A look at Figure 5.18 tells us that $t_3 = \frac{T}{2}$. So $t_3 = \frac{224.7018}{2} = 112.3509$ days. Rewriting Gauss's equation $\tan \frac{\alpha(t)}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan \frac{\beta(t)}{2}$ in the form $\tan \frac{\beta(t)}{2} = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan \frac{\alpha(t)}{2}$ and using it with $t = t_1$ and t_2 , we get

$$\tan \frac{\beta(t_1)}{2} = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan\left(\frac{1}{2} \cdot \frac{\pi}{3}\right) = \sqrt{\frac{1-0.0068}{1+0.0068}} \tan \frac{\pi}{6} = \sqrt{\frac{1-0.0068}{1+0.0068}} \frac{1}{\sqrt{3}} = 0.5734, \text{ and}$$

$$\tan \frac{\beta(t_2)}{2} = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan\left(\frac{1}{2} \cdot \frac{2\pi}{3}\right) = \sqrt{\frac{1-0.0068}{1+0.0068}} \tan \frac{\pi}{3} = \sqrt{\frac{1-0.0068}{1+0.0068}} \sqrt{3} = 1.7203.$$

By pushing the tan inverse button of a calculator (in degrees), we get $\frac{\beta(t_1)}{2} = 0.5206$ and $\frac{\beta(t_2)}{2} = 1.0442$. So $\beta(t_1) = 1.0412$ and $\beta(t_2) = 2.0885$. Inserting the term $\beta(t_1)$ into Kepler's equation $\beta(t) - \varepsilon \sin \beta(t) = \frac{2\pi t}{T}$, we get

$$\frac{2\pi t_1}{T} = \beta(t_1) - \varepsilon \sin \beta(t_1) = 1.0412 - (0.0068)(\sin 1.0412) = 1.0353,$$

so that $t_1 = \frac{(1.0353)(224.7018)}{2\pi} = 37.0248$ days. Doing so with $\beta(t_2)$, we get

$$\frac{2\pi t_2}{T} = \beta(t_2) - \varepsilon \sin \beta(t_2) = 2.0885 - (0.0068)(\sin 2.0885) = 2.0826,$$

so that $t_2 = \frac{(2.0826)(224.7018)}{2\pi} = 74.4785$ days.

So Venus traces out the first 60° of its orbit in $t_1 = 37.0248$ days, the second 60° in $t_2 - t_1 = 74.4785 - 37.0248 = 37.4537$ days, and the third 60° in $t_3 - t_2 = 112.3509 - 74.4785 = 37.8724$ days.

Problem 5.11. i. Table 5.1 informs us that for the eccentricity of ε of Mercury is $\varepsilon = 0.20563593$, so that $\varepsilon < 0.20564$. The i -th approximation β_i of $\beta(t)$ satisfies $|\beta(t) - \beta_i| < \varepsilon^i < (0.20564)^i$. Repeated squaring tells us that $\varepsilon^2 < 0.04229$, $\varepsilon^4 < 0.00179$, and $\varepsilon^8 < 0.000004$. So 8 steps should always suffice. The fact that $\varepsilon^6 < 0.00008$ and $\varepsilon^7 < 0.00002$ suggests that 6 steps will generally not be enough, but that 7 should be.

The inequality $|x + y| \leq |x| + |y|$ holds for any real numbers x and y . After applying it in the current situation, $|\beta_8 - \beta_7| = |\beta_8 - \beta(t) + \beta(t) - \beta_7| \leq |\beta_8 - \beta(t)| + |\beta(t) - \beta_7| < \varepsilon^8 + \varepsilon^7 < 0.000024$. So $|\beta_8 - \beta_7|$ rounds to zero. Hence $|\beta_7|$ and $|\beta_8|$ are equal when rounded to four decimal place accuracy. So β_7 approximates $\beta(t)$ for any t .

ii. Our computations will carry six decimal places. So from Table 5.1, the eccentricity and period of Mercury's orbit are respectively, $\varepsilon = 0.205636$ and $T = 0.240849$ years and hence $(0.240849)(365.25) = 87.970097$ days. Turn to section 5E and Kepler's equation $\beta(t) - \varepsilon \sin \beta(t) = \frac{2\pi t}{T}$. For $t = 20$ days, the successive approximation scheme $\beta_1 = \frac{2\pi t}{T}$ and $\beta_{i+1} = \frac{2\pi t}{T} + \varepsilon \sin \beta_i = \beta_1 + \varepsilon \sin \beta_i$ provides the solution

$$\begin{aligned}\beta_1 &= \frac{2\pi t}{T} = \frac{2\pi(20)}{87.970097} = 1.428482, \\ \beta_2 &= \beta_1 + \varepsilon \sin \beta_1 = 1.428482 + (0.205636) \sin(1.428482) = 1.632039 \\ \beta_3 &= \beta_1 + \varepsilon \sin \beta_2 = 1.428482 + (0.205636) \sin(1.632039) = 1.633732 \\ \beta_4 &= \beta_1 + \varepsilon \sin \beta_3 = 1.428482 + (0.205636) \sin(1.633732) = 1.633711 \\ \beta_5 &= \beta_1 + \varepsilon \sin \beta_4 = 1.428482 + (0.205636) \sin(1.633711) = 1.633711.\end{aligned}$$

of Kepler's equation for $\beta(t)$. So for $t = 20$ days, the approximation method tells us that after five iterations $\beta(20) = 1.633711$ radians (or about 93.60°).

The formula of Problem 5.8 tells us that Mercury travels the first quarter of its orbit in

$$t = \frac{T}{4} - \frac{T\varepsilon}{2\pi} = \frac{87.970097}{4} - \frac{(87.970097)(0.205636)}{2\pi} \approx 19.1134 \text{ days}$$

after perihelion. Therefore 20 days after perihelion, Mercury has traced out a little more than the first quarter of its orbit.

iii. Taking $\beta(20) = 1.633711$ in the formula $r(t) = a(1 - \varepsilon \cos \beta(t))$ of section 5B, we get $r(20) = (57,909,227)(1 - (0.205636) \cos(1.633711)) = 58,657,934$ km. To compute $\alpha(20)$ we'll suppose that Mercury is in its first post-perihelion orbit, so that the formula $\alpha(t) = 2 \tan^{-1}(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan \frac{\beta(t)}{2}) + 2\pi n_t$ applies with $n_t = 0$. It follows that $\alpha(20) = 2 \tan^{-1}(\sqrt{\frac{1+0.205636}{1-0.205636}} \tan \frac{1.633711}{2}) = 1.839085$ radians, or about 105.37 degrees.

Problem 5.13. Suppose that you have a statement S_i (mathematical or not) for each positive integer $i \geq 1$. If you know that S_1 is true and if you also know for any $i \geq 1$ that the truth of S_i implies the truth of S_{i+1} , then the statement S_i is true for all $i \geq 1$. That this is so is simple to see.

Since S_1 is true and the truth of S_1 implies the truth of S_2 , we know that S_2 is true. This in turn implies the truth of S_3 , and in turn S_4 , and in turn S_5 , etc., etc. Since this can go on forever, all statements S_i are true. This is a fact that is known as the *principle of mathematical induction*.

Consider Kepler's equation $\beta(t) - \varepsilon \sin \beta(t) = \frac{2\pi t}{T}$, with t some given elapsed time from perihelion, for some body in an elliptical orbit of eccentricity ε and perihelion orbit T . Focus on the solution $\beta(t)$ of the equation. Let $\beta_1 = \frac{2\pi t}{T}$ and for any $i \geq 1$, let $\beta_{i+1} = \frac{2\pi t}{T} + \varepsilon \sin \beta_i = \beta_1 + \varepsilon \sin \beta_i$. We have defined β_i for all $i \geq 1$. The statement S_i , for any $i \geq 1$, is the assertion $|\beta(t) - \beta_i| \leq \varepsilon^i$. We'll now verify that the conditions required for the principle of mathematical induction to hold are met by the set of statements S_i .

Since $|\beta(t) - \beta_1| = |(\frac{2\pi t}{T} + \varepsilon \sin \beta(t)) - \frac{2\pi t}{T}| = |\varepsilon \sin \beta(t)| \leq \varepsilon$, we know that $|\beta(t) - \beta_1| \leq \varepsilon^1$. So statement S_1 is true. Next, we need to *show that if S_i is true, then S_{i+1} is true as well*. So assume that it is the case that $|\beta(t) - \beta_i| \leq \varepsilon^i$. Since

$$|\beta(t) - \beta_{i+1}| = |(\frac{2\pi t}{T} + \varepsilon \sin \beta(t)) - (\beta_1 + \varepsilon \sin \beta_i)| = |\varepsilon \sin \beta(t) - \varepsilon \sin \beta_i| \leq \varepsilon |\beta(t) - \beta_i| \leq \varepsilon \cdot \varepsilon^i = \varepsilon^{i+1},$$

(note the use of the inequality $|\sin x_1 - \sin x_2| \leq |x_1 - x_2|$ of section 5E), it follows that statement S_{i+1} is true. So for any $i \geq 1$, the truth of S_i implies the truth of S_{i+1} . Since its conditions are met, the principle of mathematical induction tells us that statement S_i is true for all $i \geq 1$. Therefore, $|\beta(t) - \beta_i| \leq \varepsilon^i$ for all $i \geq 1$. Since $\varepsilon < 1$, the sequence $\beta_1, \beta_2, \beta_3, \beta_4, \dots$ converges to $\beta(t)$.

Problem 5.15. Earth's eccentricity is $\varepsilon = 0.016711$ and in the situation in question $\beta_1 = \frac{2\pi t}{T} = \frac{2\pi(170.000046)}{365.259636} = 2.924336$. The $\beta(t)$ we need is the solution x of $f(x) = x - \varepsilon \sin x - \beta_1 = 0$. The β_2 that Newton-Raphson gives us is

$$\beta_2 = \beta_1 - \frac{f(\beta_1)}{f'(\beta_1)} = \beta_1 - \frac{\beta_1 - \varepsilon \sin \beta_1 - \frac{2\pi t}{T}}{1 - \varepsilon \cos \beta_1} = 2.924336 - \frac{2.924336 - (0.016711) \sin(2.924336) - 2.924336}{1 - (0.016711) \cos(2.924336)} = 2.9278799.$$

Rounding this answer off to six decimal places gives $\beta_2 = 2.927880$. Therefore the Newton-Raphson approximation provides the correct result after a single step.

Problem 5.17. If the orbit of the point-mass P is a circle, then a is its radius, $r(t) = a$ and $\gamma(t) = \frac{\pi}{2}$ throughout. So suppose that the orbit is not a circle. If $\gamma(t_1) = \gamma(t_2) = \frac{\pi}{2}$, then P is either at periapsis or at apoapsis at times t_1 and t_2 . If P is at periapsis at both times or at apoapsis at both times, then $r(t_1) = r(t_2)$. If P is at periapsis at one of these times and at apoapsis at the other, then $r(t_1) + r(t_2) = a(1 - \varepsilon) + a(1 + \varepsilon) = 2a$. In reference to the counter clockwise motion of P the times t_1 and t_2 refer to the elapsed times after some fixed perihelion. Beyond that there are no assumptions on t_1 and t_2 . In particular, $|t_1 - t_2|$ can be greater than the period T , so that P can be in different orbits around S .

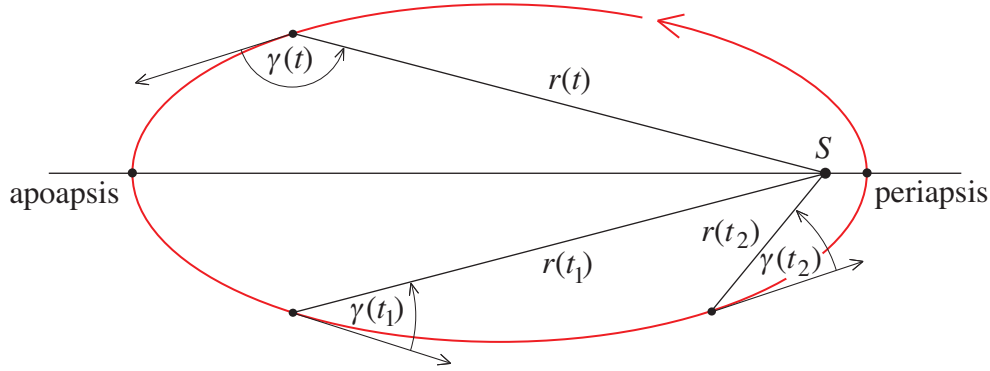
Recall from section 5D that $\sin \gamma(t) = \frac{a\sqrt{1-\varepsilon^2}}{\sqrt{r(t)(2a-r(t))}}$. Suppose that $r(t_1) = r(t_2)$ or $r(t_1) + r(t_2) = 2a$. In the first case, it is clear that $\sin \gamma(t_1) = \sin \gamma(t_2)$. In the second case, $r(t_1) = 2a - r(t_2)$ and $2a - r(t_1) = r(t_2)$. Since

$$\sin \gamma(t_1) = \frac{a\sqrt{1-\varepsilon^2}}{\sqrt{r(t_1)(2a-r(t_1))}} = \frac{a\sqrt{1-\varepsilon^2}}{\sqrt{(2a-r(t_2))r(t_2)}} = \sin \gamma(t_2),$$

$\sin \gamma(t_1) = \sin \gamma(t_2)$ in this case also. Suppose conversely, that $\sin \gamma(t_1) = \sin \gamma(t_2)$, it follows that $r(t_1)(2a - r(t_1)) = r(t_2)(2a - r(t_2))$ and hence that $a^2 - 2ar(t_1) + r(t_1)^2 = a^2 - 2ar(t_2) + r(t_2)^2$.

Therefore $(a - r(t_1))^2 = (a - r(t_2))^2$ and hence $a - r(t_1) = \pm(a - r(t_2))$. It follows that either $r(t_1) = r(t_2)$ or $r(t_1) + r(t_2) = 2a$. In particular, if $\gamma(t_1) = \gamma(t_2)$, then either $r(t_1) = r(t_2)$ or $r(t_1) + r(t_2) = 2a$.

We'll now assume that $\sin \gamma(t_1) = \sin \gamma(t_2)$ or, equivalently, that either $r(t_1) = r(t_2)$ or $r(t_1) + r(t_2) = 2a$. If P is moving from apoapsis to periapsis at both t_1 and t_2 , then $0 \leq \gamma(t) \leq \frac{\pi}{2}$ for both $t = t_1$ and $t = t_2$. Since $\sin \gamma(t_1) = \sin \gamma(t_2)$, it follows from the graph of the sine in Figure 3.4, that $\gamma(t_1) = \gamma(t_2)$. If P is moving from periapsis to apoapsis at both t_1 and t_2 , then



Sol 5.3

$\frac{\pi}{2} \leq \gamma(t) \leq \pi$ for both $t = t_1$ and $t = t_2$. Since $\sin \gamma(t_1) = \sin \gamma(t_2)$, it again follows from the graph of the sine in Figure 3.4, that $\gamma(t_1) = \gamma(t_2)$. The diagram in Sol 5.3 illustrates what is going on.

Problem 5.19. Solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for y , we get $y^2 = b^2(1 - \frac{x^2}{a^2}) = \frac{b^2}{a^2}(a^2 - x^2)$, so that the graph of the function $f(x) = \frac{b}{a}\sqrt{a^2 - x^2} = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$ is the upper half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The symmetry of this ellipse along with a standard integral formula for the length of the graph of a function, tells us that the full length of the ellipse is equal to $4 \int_0^a \sqrt{1 + f'(x)^2} dx$. Since $f'(x) = \frac{1}{2} \frac{b}{a} (a^2 - x^2)^{-\frac{1}{2}} (-2x) = -\frac{b}{a} \frac{x}{(a^2 - x^2)^{\frac{1}{2}}}$ and $a^2 - b^2 = c^2 = a^2 \varepsilon^2$, we get

$$1 + f'(x)^2 = 1 + \frac{b^2}{a^2} \frac{x^2}{a^2 - x^2} = \frac{a^2(a^2 - x^2)}{a^2(a^2 - x^2)} + \frac{b^2 x^2}{a^2(a^2 - x^2)} = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)} = \frac{a^2 - \varepsilon^2 x^2}{a^2 - x^2}.$$

With the trig substitution $x = a \sin \theta$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and hence $-a \leq x \leq a$, we get $\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta$ and $dx = a \cos \theta d\theta$, and therefore

$$4 \int_0^a \sqrt{1 + f'(x)^2} dx = 4 \int_0^a \sqrt{\frac{a^2 - \varepsilon^2 x^2}{a^2 - x^2}} dx = 4 \int_0^{\frac{\pi}{2}} \sqrt{a^2 - \varepsilon^2 a^2 \sin^2 \theta} d\theta = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \sin^2 \theta} d\theta.$$

Problem 5.21. With t varying from $t = 0$ to $t = \frac{2\pi}{\sqrt{C}}$, the average value of $\frac{p(t)}{s}$ is given by

$$\frac{\sqrt{C}}{2\pi} \int_0^{\frac{2\pi}{\sqrt{C}}} \frac{p(t)}{s} dt = \frac{2\pi}{\sqrt{C}} \int_0^{\frac{2\pi}{\sqrt{C}}} \frac{q-s}{s} \cos \sqrt{C}t dt = \frac{2\pi(q-s)}{\sqrt{C}s} \left[\frac{1}{\sqrt{C}} \sin \sqrt{C}t \right]_0^{\frac{2\pi}{\sqrt{C}}} = \frac{2\pi(q-s)}{Cs} (\sin 2\pi - \sin 0) = 0.$$

We'll conclude with a comment about the discussion in the paragraph *The Perturbing Force of an Interior Planet*. Consider the force with which a circular ring of mass M , radius R , and center S attracts a point-mass P of mass m that lies within the circle. The verification of the fact that the magnitudes of the vertical and horizontal components of this force on the point-mass P are equal to

$$\frac{GmMr}{\pi R(r^2-R^2)} \int_{-\phi_{\max}}^{\phi_{\max}} \frac{\sin \phi \cos \phi}{\sqrt{1-\frac{r^2}{R^2} \sin^2 \phi}} d\phi \quad \text{and} \quad \frac{GmMr}{\pi R(r^2-R^2)} \int_{-\phi_{\max}}^{\phi_{\max}} \frac{\cos^2 \phi}{\sqrt{1-\frac{r^2}{R^2} \sin^2 \phi}} d\phi,$$

respectively, relies on routine computations that are identical to those in part (i) of section 5I. The rest of the verification of the fact that this force on P acts in the direction of S with magnitude

$$G(r) = \frac{GmM}{(r^2-R^2)} \left[1 - \frac{1}{2^2} \left(\frac{R}{r} \right)^2 - \frac{(1 \cdot 3)^2}{(2 \cdot 4)^2} \frac{1}{3} \left(\frac{R}{r} \right)^4 - \frac{(1 \cdot 3 \cdot 5)^2}{(2 \cdot 4 \cdot 6)^2} \frac{1}{5} \left(\frac{R}{r} \right)^6 - \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \right)^2 \frac{1}{7} \left(\frac{R}{r} \right)^8 - \dots \right]$$

is carried out in the paragraph in detail.

Chapter 6. Solutions of Odd Problems

Problem 6.1. By a formula of section 6D, $\varepsilon_1 = \frac{d_1 - q_1}{d_1 + q_1} = \frac{101 - 99}{101 + 99} = \frac{2}{200} = 0.01$. By another formula from this section and the fact that GM for Eros is $4.4621 \times 10^{-4} \text{ km}^3/\text{sec}^2$, the velocity of *NEAR-Shoemaker* at apoapsis of its post OCM-4 orbit was

$$v_1 = \sqrt{\frac{GM}{d_1}} \sqrt{1 - \varepsilon_1} = \frac{\sqrt{4.4621 \times 10^{-4}}}{\sqrt{101}} \cdot \sqrt{0.99} \approx \frac{2.112 \times 10^{-2}}{10.050} (0.9950) \approx 0.00209 \text{ km/sec}$$

or 2.09 m/sec. Since the OCM-5 burn tightened the periapsis distance of *NEAR-Shoemaker's* orbit to $q_2 = 50 \text{ km}$ while keeping the apoapsis distance at $d_2 = 101 \text{ km}$, we get that the eccentricity of this orbit was $\varepsilon_2 = \frac{d_2 - q_2}{d_2 + q_2} = \frac{101 - 50}{101 + 50} = \frac{51}{151} = 0.3377$. The velocity at apoapsis of this orbit was

$$v_2 = \sqrt{\frac{GM}{d_2}} \sqrt{1 - \varepsilon_2} = \frac{\sqrt{4.4621 \times 10^{-4}}}{\sqrt{101}} \cdot \sqrt{0.6623} \approx \frac{2.112 \times 10^{-2}}{10.050} (0.8138) \approx 0.00171 \text{ km/sec}$$

or 1.71 m/sec. Alternatively, by another formula of section 6D, $\frac{v_2}{v_1} = \frac{\sqrt{1 - \varepsilon_2}}{\sqrt{1 - \varepsilon_1}} = \frac{\sqrt{1 - 0.3377}}{\sqrt{1 - 0.01}} = \frac{0.8138}{0.9950} = 0.8179$. So as before, $v_2 = 0.8179 v_1 = (0.8179)(2.09) \approx 1.71 \text{ m/sec}$.

Problem 6.3. The problem lists the periapsis velocities incorrectly as 3.25 m/sec and 3.07 m/sec, respectively. The correct values are 3.89 m/sec and 3.67 m/sec, respectively. The difference $3.89 - 3.67 = 0.22 \text{ m/sec}$ (not 0.18 m/sec as incorrectly stated in the problem) is in close agreement with the $\Delta v = 0.24 \text{ m/sec}$ of the table. Using formulas of section 6D with $q_1 = 35$ and $d_1 = 51$, we know that the eccentricity of the orbit of *NEAR-Shoemaker* just prior to OCM-8 was $\varepsilon_1 = \frac{d_1 - q_1}{d_1 + q_1} = \frac{51 - 35}{51 + 35} = \frac{16}{86} = 0.1860$ and that the speed at periapsis in this orbit was

$$v_1 = \sqrt{\frac{GM}{q_1}} \sqrt{1 + \varepsilon_1} = \frac{\sqrt{4.4621 \times 10^{-4}}}{\sqrt{35}} \cdot \sqrt{1.1860} \approx \frac{2.112 \times 10^{-2}}{5.9161} (1.0890) \approx 0.00389 \text{ km/sec}$$

or 3.89 m/sec. Since the OCM-8 burn tightened the apoapsis distance of *NEAR-Shoemaker's* orbit to $d_2 = 39 \text{ km}$ while keeping the periapsis distance at $q_2 = 35 \text{ km}$, we get that the eccentricity of the post OCM-8 orbit was $\varepsilon_2 = \frac{d_2 - q_2}{d_2 + q_2} = \frac{39 - 35}{39 + 35} = \frac{4}{74} = 0.0545$ and that the spacecraft's velocity at periapsis of this new orbit was

$$v_2 = \sqrt{\frac{GM}{q_2}} \sqrt{1 + \varepsilon_2} = \frac{\sqrt{4.4621 \times 10^{-4}}}{\sqrt{35}} \cdot \sqrt{1.0545} \approx \frac{2.112 \times 10^{-2}}{5.9161} (1.0269) \approx 0.00367 \text{ km/sec}$$

or 3.67 m/sec.

Problem 6.5. To start, refer to Figure 6.26 and check the details of the discussion of the spacecraft's elliptical transfer orbit from the insertion point I to the rendezvous point P_1 . Now turn to the Hohmann transfer from I to P_2 . Let t_2 be the time required for this transfer and let $\alpha(t_2) = 100^\circ$ and $r(t_2) = 2.0700 \times 10^8 \text{ km}$ be the corresponding angle and distance for the rendezvous. Let a_2 and ε_2 be the semimajor axis and eccentricity of the transfer to P_2 . Note that the periapsis distance $a_2(1 - \varepsilon_2)$ is the same $1.4710 \times 10^8 \text{ km}$ as before. After inserting the values $\cos \alpha(t_2) = \cos(100) = -0.1736$ and $r(t_2) = 2.0700 \times 10^8$ into the formula

$$r(t_2) = \frac{a_2(1 - \varepsilon_2^2)}{1 + \varepsilon_2 \cos \alpha(t_2)} = \frac{a_2(1 - \varepsilon_2)(1 + \varepsilon_2)}{1 + \varepsilon_2 \cos \alpha(t_2)},$$

we get $2.0700 \times 10^8 = \frac{(1.4710 \times 10^8)(1 + \varepsilon_2)}{1 + \varepsilon_2(-0.1736)}$, and hence $(2.0700)(1 - 0.1736\varepsilon_2) = (1.4710)(1 + \varepsilon_2)$. Therefore $(1.4710 + 0.1736(2.0700))\varepsilon_2 = 2.0700 - 1.4710$, so that

$$\varepsilon_2 = \frac{2.0700 - 1.4710}{1.4710 + 0.1736(2.0700)} = \frac{0.5990}{1.8304} = 0.3273.$$

This tells us that $0.6727a_2 = a_2(1 - \varepsilon_2) = 1.4710 \times 10^8$ km and hence that $a_2 = 2.1867 \times 10^8$ km. Since the insertion point I of the craft into its solar transfer orbit is the perihelion of the orbit and since $GM \approx 1.3271 \times 10^{20} \text{ m}^3/\text{sec}^2 = 1.3271 \times 10^{11} \text{ km}^3/\text{sec}^2$ for the Sun, Example 5.1 applies to tell us that the craft needs to be inserted into its transfer with a velocity of

$$v_2 = \sqrt{\frac{GM(1+\varepsilon_2)}{a_2(1-\varepsilon_2)}} \approx \sqrt{\frac{1.3271 \times 10^{11}(1+0.3273)}{1.4710 \times 10^8}} \approx \sqrt{\frac{(1.3271)(1.3273) \times 10^3}{1.4710}} \approx 34.604 \text{ km/sec}.$$

The computation of t_2 remains. By solving Gauss's equation of Chapter 5B for $\tan \frac{\beta(t_2)}{2}$, we get $\tan \frac{\beta(t_2)}{2} = \sqrt{\frac{1-\varepsilon_2}{1+\varepsilon_2}} \tan \frac{\alpha(t_2)}{2}$. Since $\alpha(t_2) = 100(\frac{\pi}{180}) = \frac{10}{18}\pi$ radians, it follows that

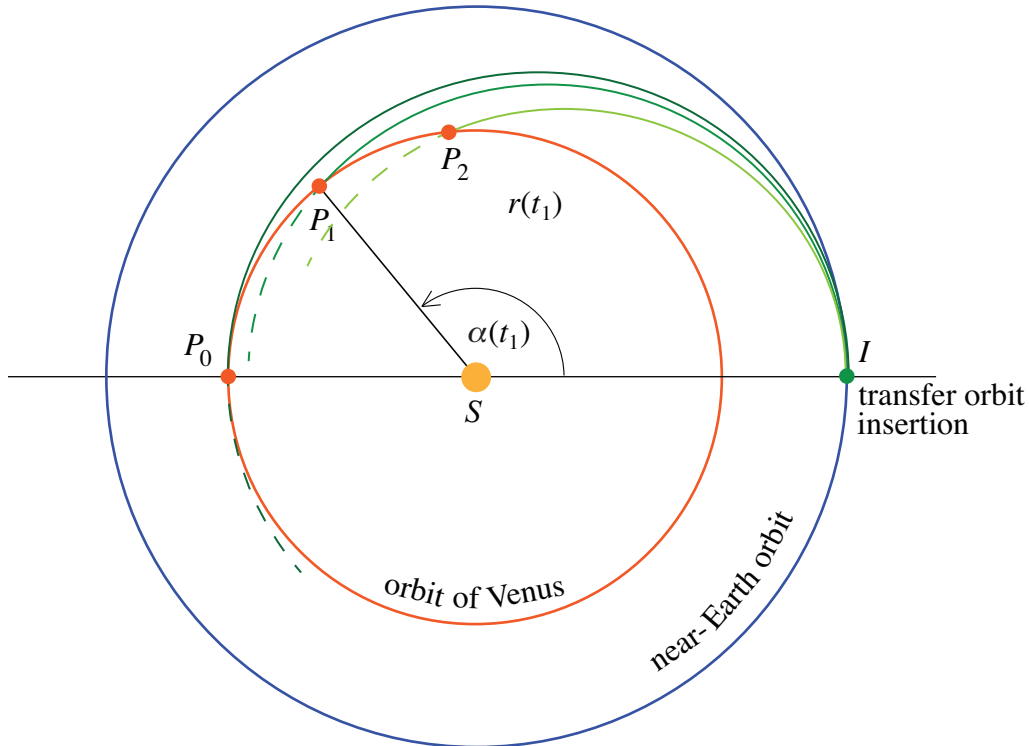
$$\tan \frac{\beta(t_2)}{2} \approx \sqrt{\frac{1-0.3273}{1+0.3273}} \tan(\frac{5}{18}\pi) \approx 0.8484.$$

So $\frac{\beta(t_2)}{2} \approx \tan^{-1}(0.8484)$ and $\beta(t_2) \approx 1.4071$. Solving Kepler's equation $\beta(t_2) - \varepsilon_2 \sin \beta(t_2) = \sqrt{\frac{GM}{a_2^3}} t_2$ of Chapter 5C for t_2 and inserting the values we have, we get

$$t_2 = \sqrt{\frac{a_2^3}{GM}} (\beta(t_2) - \varepsilon_2 \sin \beta(t_2)) \approx \sqrt{\frac{(2.1867 \times 10^8)^3}{1.3271 \times 10^{11}}} (1.4071 - 0.3273 \sin(1.4071)) \approx 9,623,500 \text{ sec}$$

or about 111 days.

Problem 6.7. The diagram of Sol 6.1 below sketches the orbits of a spacecraft's near Earth solar orbit, the orbit of Venus, as well as three elliptical solar transfer orbits (in shades of green) (in shades of green)



Sol 6.1

from the insertion point I to the rendezvous points P_0, P_1 , and P_2 , respectively.

Problem 6.9. We'll verify the formulas for v_{in} and v_{out} in the situation of Figure 6.29. The various velocity vectors have the same meaning as in section 6J. The velocity of the planet P relative to S is represented by the vector v_P that has length the speed of P and direction given by the tangent to its orbital path. The figure depicts the vector v_P and the angle θ_P between it and the vector v_∞ of the craft's entry into the planet's gravitational sphere of influence (SOI). As before, we'll assume that the velocity vector v_P of P relative to S is constant. Figure 6.29 also depicts the velocity vector v_∞ at the point of the craft's departure from the planet's SOI. The respective resultants of v_P and the two vectors v_∞ , each determined by the parallelogram law, are drawn into the figure as v_{in} and v_{out} in blue and green, respectively. The vector v_{in} represents the velocity of the craft C relative to the Sun at the point of entry into the SOI and the vector v_{out} the velocity of the craft relative to the Sun at the point of departure from the SOI. We know that the angle of deflection δ is equal to $\delta = 2 \sin^{-1}(\frac{1}{\varepsilon})$. The magnitudes of v_{in} and v_{out} as well as the angle between them can be computed by making use of basic trigonometry. The computation of the magnitudes relies on Figure 6.30 (extracted from Figure 6.29). The law of cosines applied to the triangle that the vectors v_P and v_{in} determine, tells us that (now also using $v_\infty, v_P, v_{\text{in}}$, and v_{out} for the magnitudes of these vectors) $v_{\text{in}}^2 = v_\infty^2 + v_P^2 - 2v_\infty v_P \cos \theta_P$ and hence that

$$v_{\text{in}} = \sqrt{v_\infty^2 + v_P^2 - 2v_\infty v_P \cos \theta_P}.$$

By the law of cosines applied to the triangle formed by the vectors v_P and v_{out} , we get $v_{\text{out}}^2 = v_\infty^2 + v_P^2 - 2v_\infty v_P \cos(\theta_P + \delta)$. It follows that

$$v_{\text{out}} = \sqrt{v_\infty^2 + v_P^2 - 2v_\infty v_P \cos(\theta_P + \delta)}.$$

Problem 6.11. We turn to *Voyager 2*'s flyby of Saturn. Let P designate the planet Saturn and turn to Tables 6.5 and 6.6 for the needed data. For M_P the mass of Saturn, $GM_P = 37,940,585 \text{ km}^3/\text{sec}^2$. The Sun-relative speed of Saturn at the time of the flyby was $v_P = 9.59 \text{ km/sec}$ and the angle between Saturn's Sun-relative velocity and the direction of *Voyager 2*'s approach to Saturn (as described in Figure 6.17) was $\theta_P = 98.2^\circ$. Using data in Table 6.5, we get that

$$v_\infty = \sqrt{\frac{GM_P}{a}} = \sqrt{\frac{37,940,585}{332,965}} \approx 10.67 \text{ km/sec}$$

for *Voyager 2*'s flyby and $a(\varepsilon - 1) = 332,965(1.482601 - 1) = 160,689 \text{ km}$ for the distance of the craft from Saturn's center of mass at periapsis of the flyby. Since $\varepsilon = 1.482601$, we get $\delta = 2 \sin^{-1}(\frac{1}{\varepsilon}) = 2 \sin^{-1}(\frac{1}{1.482601}) = 84.83^\circ$ for the angle of deflection. It follows that

$$\begin{aligned} v_{\text{in}} &= \sqrt{v_\infty^2 + v_P^2 - 2v_\infty v_P \cos \theta_P} \approx \sqrt{10.67^2 + 9.59^2 - 2(10.67)(9.59) \cos 98.2} \\ &\approx 15.33 \text{ km/sec, and} \end{aligned}$$

$$\begin{aligned} v_{\text{out}} &= \sqrt{v_\infty^2 + v_P^2 - 2v_\infty v_P \cos(\theta_P + \delta)} \approx \sqrt{10.67^2 + 9.59^2 - 2(10.67)(9.59) \cos(98.2 + 84.83)} \\ &\approx 20.25 \text{ km/sec.} \end{aligned}$$

Voyager 2's flyby of Saturn resulted in a change of direction of

$$\begin{aligned}
\varphi &= \varphi_1 + \varphi_2 = \sin^{-1}\left(\frac{v_\infty}{v_{\text{in}}} \sin \theta_P\right) - \sin^{-1}\left(\frac{v_\infty}{v_{\text{out}}} \sin(\theta_P + \delta)\right) \\
&\approx \sin^{-1}\left(\frac{10.67}{15.33} \sin 98.2\right) - \sin^{-1}\left(\frac{10.67}{20.25} \sin(98.2 + 84.83)\right) \\
&\approx 43.54^\circ - (-1.60^\circ) = 45.14^\circ.
\end{aligned}$$

We'll consider *Voyager 2*'s flyby of Uranus next. Let P designate Uranus and turn to Tables 6.5 and 6.6 for the needed data. For M_P the mass of Uranus, $GM_P = 5,794,585 \text{ km}^3/\text{sec}^2$. The Sun-relative speed of Uranus at the time of the flyby was $v_P = 6.71 \text{ km/sec}$ and the angle between the Sun-relative velocity of Uranus and the direction of *Voyager 2*'s approach to Uranus was $\theta_P = 106.0^\circ$. Using data in Table 6.5, we get that

$$v_\infty = \sqrt{\frac{GM_P}{a}} = \sqrt{\frac{5,794,585}{26,694}} \approx 14.73 \text{ km/sec}$$

for *Voyager 2*'s flyby and $a(\varepsilon - 1) = 26,694(5.014153 - 1) = 107,154 \text{ km}$ for the distance of the craft from the center of mass of Uranus at periapsis of the flyby. Since $\varepsilon = 5.014153$, we get $\delta = 2 \sin^{-1}(\frac{1}{\varepsilon}) = 2 \sin^{-1}(\frac{1}{5.014153}) = 23.01^\circ$ for the angle of deflection. It follows that

$$\begin{aligned}
v_{\text{in}} &= \sqrt{v_\infty^2 + v_P^2 - 2v_\infty v_P \cos \theta_P} \approx \sqrt{14.73^2 + 6.71^2 - 2(14.73)(6.71) \cos 106.0} \\
&\approx 17.79 \text{ km/sec, and}
\end{aligned}$$

$$\begin{aligned}
v_{\text{out}} &= \sqrt{v_\infty^2 + v_P^2 - 2v_\infty v_P \cos(\theta_P + \delta)} \approx \sqrt{14.73^2 + 6.71^2 - 2(14.73)(6.71) \cos(106.0 + 23.01)} \\
&\approx 19.66 \text{ km/sec.}
\end{aligned}$$

Voyager 2's flyby of Uranus resulted in a change of direction of

$$\begin{aligned}
\varphi &= \varphi_1 + \varphi_2 = \sin^{-1}\left(\frac{v_\infty}{v_{\text{in}}} \sin \theta_P\right) - \sin^{-1}\left(\frac{v_\infty}{v_{\text{out}}} \sin(\theta_P + \delta)\right) \\
&\approx \sin^{-1}\left(\frac{14.73}{17.79} \sin 106.0\right) - \sin^{-1}\left(\frac{14.73}{19.66} \sin(106.0 + 23.01)\right) \\
&\approx 52.74^\circ - 35.60^\circ = 17.14^\circ
\end{aligned}$$

(Notice that the direction changes 45.14° for the Saturn flyby and 17.14° for the Uranus flyby derived above differ from the 45.28° and 17.31° announced in the statement of Problem 6.11. The reason? The computations of the announced changes were rounded off differently.)

Problem 6.13. Solving $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for y we get, $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{1}{a^2}(x^2 - a^2)$, so that $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ and $y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$. It follows that $f(x) = \frac{b}{a} \sqrt{x^2 - a^2}$ is a function that has the upper half of the hyperbola as graph. (Note that the coefficient $\frac{b}{a}$ was erroneously omitted in the first line of the statement of the problem.) Consider Figure 6.12 and 6.9. Revolving the diagram of Figure 6.12 around the y -axis moves $(-a, 0)$ to $(a, 0)$, $(-a \cosh \beta(t), 0)$ to $(a \cosh \beta(t), 0)$, and the left branch of the graph of $f(x) = \frac{b}{a} \sqrt{x^2 - a^2}$ to the right branch. The formula

$$B(t) = -\frac{1}{2}x(t)y(t) - \int_a^{a \cosh \beta(t)} \frac{b}{a} \sqrt{x^2 - a^2} dx.$$

is a direct consequence (note that $x(t) < 0$). The formula $\cosh^{-1}u = \ln(u + \sqrt{u^2 - 1})$, where $u \geq 1$ can be found in standard calculus texts, for example in reference 2 for Chapter 3. (It is not hard to derive. Let $v = \cosh^{-1}u$ and note that $u = \cosh v = \frac{e^v + e^{-v}}{2}$, so that $e^v - 2u + e^{-v} = 0$, and

hence $(e^v)^2 - 2ue^v + 1 = 0$. Now use the quadratic formula to solve for e^v and apply the natural log function \ln .) Substituting $u = \frac{x}{a}$ with $x \geq a$ into the formula, we get

$$\cosh^{-1} \frac{x}{a} = \ln\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right) = \ln\left(\frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2}\right) = \ln \frac{1}{a}(x + \sqrt{x^2 - a^2}) = \ln \frac{1}{a} + \ln(x + \sqrt{x^2 - a^2}).$$

Notice that with $x = a$, we see that $\cosh^{-1}(1) = \ln(1 + 0) = \ln 1 = 0$.

Inserting this expression for $\cosh^{-1} \frac{x}{a}$ into the formula for $\int \sqrt{x^2 - a^2} dx$ tells us that

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln(x + \sqrt{x^2 - a^2}) + C = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}(\cosh^{-1} \frac{x}{a} - \ln \frac{1}{a}) + C.$$

After absorbing $\ln \frac{1}{a}$ into the constant C ,

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a} + C.$$

Since $\cosh^{-1}(1) = 0$, it follows that

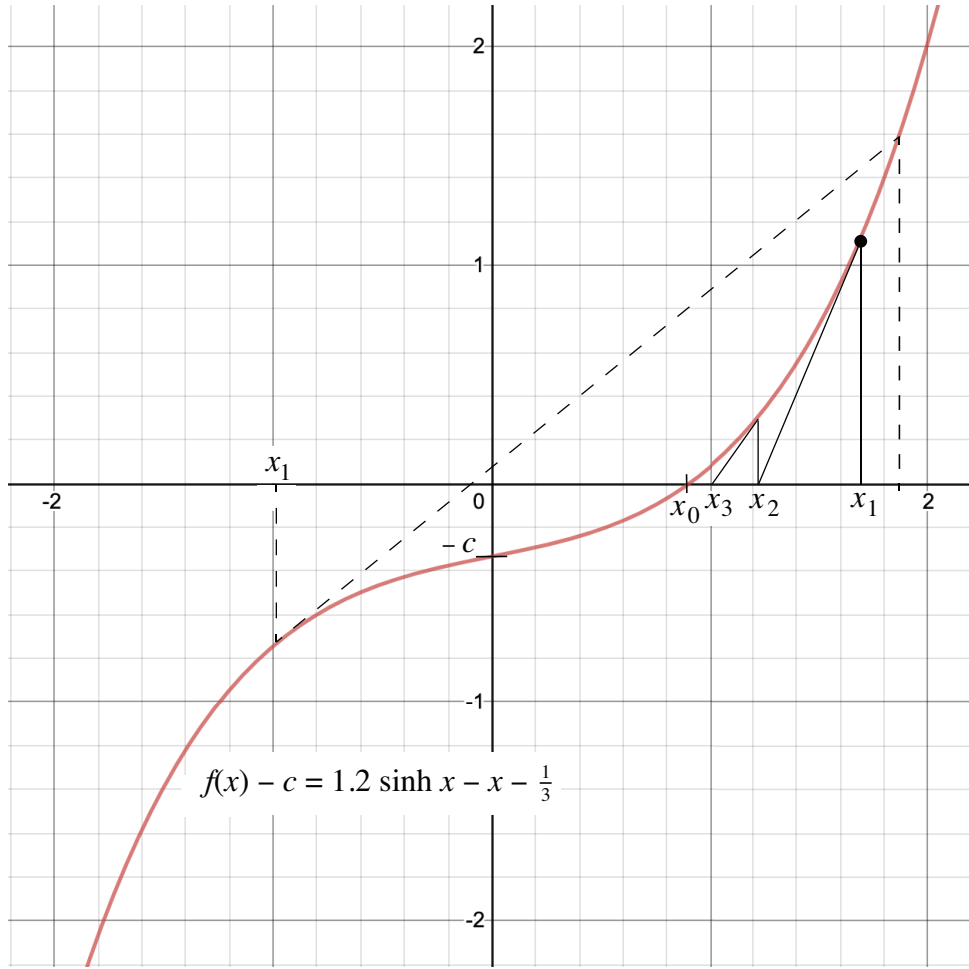
$$\begin{aligned} \int_a^{a \cosh \beta(t)} \frac{b}{a} \sqrt{x^2 - a^2} dx &= \frac{b}{a} \left(\frac{1}{2} (a \cosh \beta(t)) \sqrt{a^2 \cosh^2 \beta(t) - a^2} - \frac{a^2}{2} \cosh^{-1}(\cosh \beta(t)) \right) \\ &= \frac{1}{2} (a \cosh \beta(t)) (b \sinh \beta(t)) - \frac{1}{2} ab \beta(t) = \frac{1}{2} x(t)y(t) - \frac{1}{2} ab \beta(t), \end{aligned}$$

Therefore $B(t) = \frac{1}{2} ab \beta(t)$.

Problem 6.15. Superimpose a polar coordinate system over the Cartesian system of Figure 6.31 as described in Chapter 3C. It follows from the connection between the polar and Cartesian systems developed there, in particular by the discussion of Figure 3.13, that $P = (a \cos \beta, a \sin \beta)$ for some positive real number β . The polar equation of the circle is $r = f(\theta) = a$, so that by the polar area integral of Chapter 3G, the highlighted sector has area $B = \int_0^\beta \frac{1}{2} a^2 d\theta = \frac{1}{2} a^2 \theta \Big|_0^\beta = \frac{1}{2} a^2 \beta$.

Problem 6.17. We are given that $y = f(x)$ is a differentiable function that satisfies $f(0) = 0$, $f'(x) > 0$ for $x \neq 0$, and that the graph of $y = f(x)$ is concave up for $x \geq 0$ and concave down for $x \leq 0$. Consider $y = f(x) - c$ with $c \geq 0$ a constant and suppose that $f(x_0) - c = 0$. Since $f(0) - c = -c < 0$, and $f(x_0) - c = 0$, it follows that $x_0 \geq 0$.

Consider the tangent line to the graph of $y = f(x) - c$ at the point $(x_1, f(x_1) - c)$. The point-slope form of the equation of this line is $y - (f(x_1) - c) = f'(x_1)(x - x_1)$. Let x_2 be the x -coordinate of the point of intersection of the line with the x -axis. Since $y = 0$ at that point, $-(f(x_1) - c) = f'(x_1)(x_2 - x_1)$. It follows that $x_2 = x_1 - \frac{f(x_1) - c}{f'(x_1)}$. The transition from x_1 to x_2 is illustrated in Sol 6.3 in the special case of $f(x) = 1.2 \sinh x - x$ and $c = \frac{1}{3}$. Repeating the procedure that led from x_1 to x_2 , with x_2 gives rise to x_3 , then to x_4 , and so on. Sol 6.3 shows how and why the sequence $x_1, x_2, x_3, \dots, x_i, \dots$, where for each x_i , $x_{i+1} = x_i - \frac{f(x_i) - c}{f'(x_i)}$, converges to the x_0 that satisfies $f(x_0) - c = 0$. If the initial stab x_1 at a solution is to the left of x_0 , again see Sol 6.3, then the succeeding x_2 falls to the right and we're back in the earlier situation. The graph of Sol 6.3 illustrates the recipe: from any approximation—this is a point on the x -axis—starting with x_1 , go vertically to the graph and slide down (or up) the tangent back to x -axis to get the next



Sol 6.3. The graph of $f(x) = 1.2 \sinh x - x - \frac{1}{3}$ as drawn with <https://www.desmos.com/calculator>.

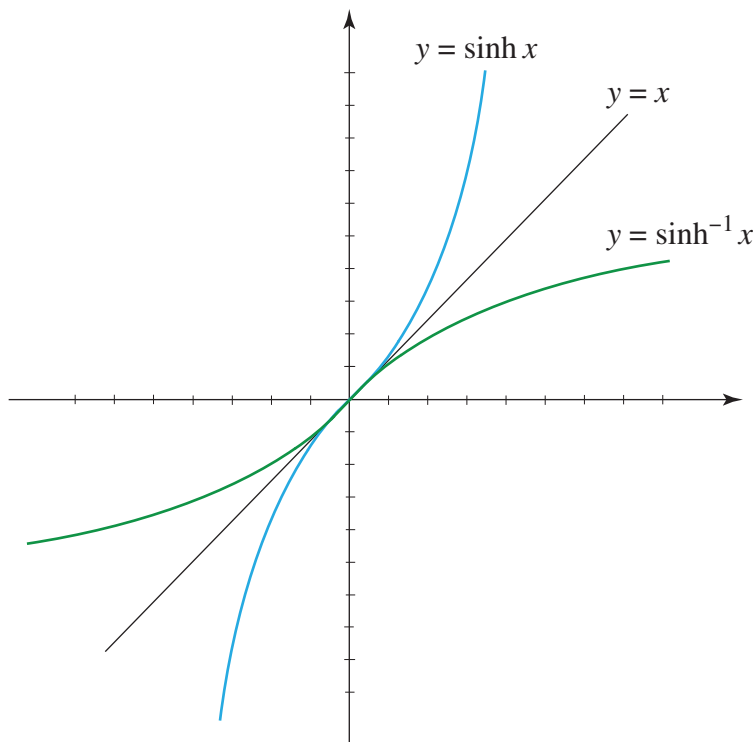
approximation. (If $f'(0) = 0$, then $x = 0$ is problematic in this regard, because the tangent line at $(0, -c)$ is parallel to the x -axis and does not intersect it.)

Consider the hyperbolic Kepler equation $\varepsilon \sinh x - x = \sqrt{\frac{GM}{a^3}}t$, where $t \geq 0$ is a constant.

What was just set out applies with $y = f(x) = \varepsilon \sinh x - x$, $c = \sqrt{\frac{GM}{a^3}}t$, and $x_0 = \beta(t)$. The facts developed in section 6E and a look at the graphs of Figure 6.10 confirm that $f(x)$ satisfies all the requirements set out in the problem: $f(0) = 0$, $f'(x) = \varepsilon \cosh x - 1 \geq \varepsilon - 1 > 0$, and since $f''(x) = \varepsilon \sinh x$, $f(0) = 0$, $f'(x) > 0$ for $x \neq 0$, the graph of $f(x)$ is concave up for $x \geq 0$ and concave down for $x \leq 0$. With $\beta_1 = \sinh^{-1} \frac{1}{\varepsilon} \left(\sqrt{\frac{GM}{a^3}}t \right)$ as a good first stab at $\beta(t)$ and

$\beta_{i+1} = \beta_i - \frac{(\varepsilon \sinh \beta_i - \beta_i) - \sqrt{\frac{GM}{a^3}}t}{\varepsilon \cosh \beta_i - 1}$, the sequence β_1, β_2, \dots , converges to $x_0 = \beta(t)$. Sol 6.3 considered the representative case $y = f(x) = 1.2 \sinh x - x$ with $c = \frac{1}{3}$.

Problem 6.19. The diagram of Sol 6.4 shows how the graph of the function $f(x) = \sinh^{-1} x$ (in green) is obtained by reflecting the the graph of $y = \sinh x$ (in blue) across the line $y = x$. Turn to Figure 6.33. Let $P_1 = (x_1, y_1) = (x_1, \sinh^{-1} x_1)$ and let L be the tangent to the graph of $y = \sinh^{-1} x$ at P_1 . Because $f'(x) = \frac{1}{\sqrt{x^2 + 1}}$ (by Example 6.10), the slope of L is $\frac{1}{\sqrt{x_1^2 + 1}}$. Therefore the point-slope



Sol 6.4

form of the equation of the tangent is $y = \frac{1}{\sqrt{x_1^2+1}}(x - x_1) + \sinh^{-1} x_1$. Since $f'(x) = (x^2 + 1)^{-\frac{1}{2}}$,

$$f''(x) = -\frac{1}{2}(x^2 + 1)^{-\frac{3}{2}}(2x) = -\frac{x}{(x^2+1)^{\frac{3}{2}}}.$$

Since $f''(x) < 0$ for $x \geq 0$, the graph of $f(x) = \sinh^{-1} x$ is concave down for $x \geq 0$. Let x_2 satisfy $x_2 > x_1$ and let $y_2 = \frac{1}{\sqrt{x_1^2+1}}(x_2 - x_1) + \sinh^{-1} x_1$ be the y -coordinate of the point on L with x -coordinate x_2 . A look at Figure 6.33 tells us that

$$|\sinh^{-1} x_2 - \sinh^{-1} x_1| < |y_2 - y_1| = \left| \frac{1}{\sqrt{x_1^2+1}}(x_2 - x_1) + \sinh^{-1} x_1 - \sinh^{-1} x_1 \right| = \frac{1}{\sqrt{x_1^2+1}}|x_2 - x_1|.$$

Refer to the solution of the hyperbolic Kepler equation of section 6H and note that the inequality above is stronger than the inequality $|\sinh^{-1} x_2 - \sinh^{-1} x_1| < |x_2 - x_1|$ (when $x_1 \neq 0$). This means that the sequence $\beta_1, \beta_2, \beta_3, \dots$ should in general converge to the solution $\beta(t)$ of Kepler's equation more quickly than the inequality $|\beta(t) - \beta_i| \leq \frac{1}{\varepsilon^i} |\beta(t)|$ suggests.

We'll look at **Problems 6.20, 6.21, and 6.22** together. To understand the printouts of the HORIZONS ephemerides system for the changing parameters of *Cassini's* trajectory, one needs to know the following abbreviations as well as the concepts to which they refer. Several of them, e.g., eccentricity, periapsis distance, semimajor axis, ... have been used throughout the text, others are illustrated in Figure 6.25.

EC = eccentricity of the trajectory, denoted by ε in this text.

QR = periapsis distance, often denoted by q in this text.

- IN = angle of inclination of the orbit relative to Earth’s orbital plane as reference plane, denoted by i in Figure 6.25.
- OM = longitude of the ascending node, the angle denoted by Ω in Figure 6.25.
- W = argument of periapsis, the angle denoted by ω in Figure 6.25. Referred to as “argument of perifocus” in HORIZONS.
- Tp = time of periapsis (in Julian day number).
- N = mean or average motion in degrees/day.
- MA = mean anomaly, the angle $\sqrt{\frac{GM}{a^3}}t$ in radians, where GM is the gravitational constant of the attracting body, a the semimajor axis of the orbit, and t the elapsed time from periapsis.
- TA = true anomaly, the angle denoted by $\alpha = \alpha(t)$ in this text, where t is the elapsed time from some periapsis. Often denoted by ν in the literature.
- A = semimajor axis. Listed with negative numbers in HORIZONS in hyperbolic situations.
- AD = apoapsis distance. Not defined, or infinite, in hyperbolic situations. When users found this problematic in some computer applications, HORIZONS set it equal to the huge, artificial number 9.9999999×10^{99} km or its equivalent $6.845864538097 \times 10^{91}$ au. (Given the estimate 8.8×10^{23} km for the diameter of the universe, this is “essentially infinite.”)
- PR = sidereal period.

The notation E−05, E+00, E+6, . . . , E+91, E+99 in HORIZONS following a number tells us that the number is multiplied by 10^{-5} , $10^0 = 1$, 10^6 , . . . or (see the discussion of AD) 10^{91} or 10^{99} .

Go to the website <https://ssd.jpl.nasa.gov/horizons.cgi> and consider **Current Settings**. Follow the instructions: Under **Ephemeris Type** [change], click on **change**, Select **Orbital Elements**, and click on **Use Selection Above**. Under **Target Body** [change], click on **change** and type **Cassini** into the box **Lookup the specified body**, then **Search**, and Select **MB: Cassini (spacecraft)**, and click on **Select Indicated Body**. Under **Center** [change] click on **change** and type **@Saturn** into the box **Specify Center**, then **Search**, Select **Saturn (body center)**, and click **Use Selected Location**. Relevant **Time Span** settings follow in the problems below. For **Table Settings** and **Display/Output** use the *defaults*. For much more information about HORIZONS the reader is directed to https://ssd.jpl.nasa.gov/?horizons_doc.

Continue with the instructions under **Current Settings**. For **Time Span** [change], click on **change**. Under **Start Time** insert 2004-July-01 01:12, under **Stop Time** insert 2004-July-01 02:48 for the SOI (or the time spans for OCM-2 or the flyby of Titan), and under **Step Size** insert 1 and **minutes**. Then click **Use Specified Time**. Finally click **Generate Ephemeris**. The table that HORIZONS generates provides the data for a minute by minute picture of *Cassini*’s changing trajectory during the SOI (or OCM-2 or the flyby of Titan).

The tables Horizons 6.20, Horizons 6.21, and Horizons 6.22 are copies of the output of the data for the Saturn-specific trajectory of *Cassini* that HORIZONS generates for each of the three maneuvers. To limit the output to a single page, the data is presented in 5 or 10 minute intervals rather than in 1 minute intervals that the problems prescribe.

Let’s start with the SOI and Table 6.20. The flow of numbers that describe *Cassini*’s evolving hyperbolic approach to Saturn tell us that the eccentricities EC decreased, so that the hyperbolas

tightened (refer to the conclusions of Problem 6.8), that its periapsis distances QR decreased slightly, while the semimajor axes A increased. Finally, during the last part of the maneuver, the trajectory flipped from tightly hyperbolic (EC slightly greater than 1) to tightly elliptical (EC slightly less than 1). A comparison of the last set of data in Table 6.20 with the first set of Table 6.21 tells us

```

2453187.550000000 = A.D. 2004-Jul-01 01:12:00.0000 TDB
EC= 1.059697417226318E+00 QR= 5.411274602285721E-04 IN= 2.477328478140550E+01
OM= 2.081463420653390E+02 W = 1.701640876514041E+02 Tp= 2453187.609464457259
N = 1.930769973808551E+01 MA=-1.148121885951914E+00 TA= 2.826299356210839E+02
A =-9.064503715078815E-03 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.556944444 = A.D. 2004-Jul-01 01:22:00.0000 TDB
EC= 1.054482833518523E+00 QR= 5.402046873659240E-04 IN= 2.477234044257927E+01
OM= 2.081697094194985E+02 W = 1.703701782873823E+02 Tp= 2453187.609640871175
N = 1.687712978043474E+01 MA=-8.893644377849712E-01 TA= 2.874630369648864E+02
A =-9.915135694663624E-03 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.563888889 = A.D. 2004-Jul-01 01:32:00.0000 TDB
EC= 1.048423782881196E+00 QR= 5.393225138840688E-04 IN= 2.477073093535841E+01
OM= 2.081904310665608E+02 W = 1.705928809484825E+02 Tp= 2453187.609828569461
N = 1.417626517903114E+01 MA=-6.512530931927840E-01 TA= 2.930557698317228E+02
A =-1.113755435438114E-02 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.570833333 = A.D. 2004-Jul-01 01:42:00.0000 TDB
EC= 1.042159676973305E+00 QR= 5.385614769016572E-04 IN= 2.476929089116380E+01
OM= 2.082020776720520E+02 W = 1.708128071181069E+02 Tp= 2453187.610004346352
N = 1.154089706081723E+01 MA=-4.520686317285711E-01 TA= 2.995850540228736E+02
A =-1.277432645517337E-02 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.577777778 = A.D. 2004-Jul-01 01:52:00.0000 TDB
EC= 1.035645553276879E+00 QR= 5.379007332762932E-04 IN= 2.476949265072492E+01
OM= 2.082014333075182E+02 W = 1.710356191542869E+02 Tp= 2453187.610168869607
N = 8.988793818510301E+00 MA=-2.911568477386485E-01 TA= 3.072117602179807E+02
A =-1.509026186515077E-02 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.584722222 = A.D. 2004-Jul-01 02:02:00.0000 TDB
EC= 1.028799750976527E+00 QR= 5.373309695591188E-04 IN= 2.477384036324965E+01
OM= 2.081857732125626E+02 W = 1.712663571700701E+02 Tp= 2453187.610323349480
N = 6.538330562710874E+00 MA=-1.673886320449780E-01 TA= 3.160835814120056E+02
A =-1.865748665664056E-02 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.591666667 = A.D. 2004-Jul-01 02:12:00.0000 TDB
EC= 1.021502172450149E+00 QR= 5.368614505233062E-04 IN= 2.478587099037905E+01
OM= 2.081549801798574E+02 W = 1.715052253738185E+02 Tp= 2453187.610467951745
N = 4.223541477280539E+00 MA=-7.940800660722776E-02 TA= 3.262862715164805E+02
A =-2.496777717544487E-02 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.598611111 = A.D. 2004-Jul-01 02:22:00.0000 TDB
EC= 1.013630205446138E+00 QR= 5.365187475607448E-04 IN= 2.480910966606691E+01
OM= 2.081146211046503E+02 W = 1.717415799320598E+02 Tp= 2453187.610599209554
N = 2.133648294937244E+00 MA=-2.557838498058706E-02 TA= 3.377678040710929E+02
A =-3.936248427662157E-02 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.605555556 = A.D. 2004-Jul-01 02:32:00.0000 TDB
EC= 1.005133362735016E+00 QR= 5.363290261436690E-04 IN= 2.484447056454429E+01
OM= 2.080771634276168E+02 W = 1.719515898597467E+02 Tp= 2453187.610708504450
N = 4.934038809415852E-01 MA=-2.542484812669823E-03 TA= 3.502585464881764E+02
A =-1.044790820031561E-01 AD= 6.684586453809735E+91 PR= 1.157407291666667E+95
2453187.612500000 = A.D. 2004-Jul-01 02:42:00.0000 TDB
EC= 9.960730080131481E-01 QR= 5.362928755212837E-04 IN= 2.488761103649933E+01
OM= 2.080568870839181E+02 W = 1.721063017195110E+02 Tp= 2453187.610784029122
N = 3.301669247024165E-01 MA= 5.665567740087296E-04 TA= 3.250940254321176E+00
A = 1.365658186512540E-01 AD= 2.725953444269868E-01 PR= 1.090357552696935E+03

```

Table 6.20. Orbital ephemerides for the insertion of *Cassini* into orbit around Saturn
(Saturn Orbit Insertion or SOI) on July 1st in 10 minute intervals.

2453241.166666667 = A.D. 2004-Aug-23 16:00:00.0000 TDB
 EC= 9.766368033090591E-01 QR= 7.195997204487096E-04 IN= 2.593808814433135E+01
 OM= 2.077595078647697E+02 W = 1.728962228176763E+02 Tp= 2453187.218644964974
 N = 3.082538448058821E+00 MA= 1.662968510915603E+02 TA= 1.792444794972650E+02
 A = 3.080056766066363E-02 AD= 6.088153560087855E-02 PR= 1.167868644839464E+02

2453241.170138889 = A.D. 2004-Aug-23 16:05:00.0000 TDB
 EC= 9.708857828327798E-01 QR= 8.992094405481015E-04 IN= 2.649713901158049E+01
 OM= 2.075865819125332E+02 W = 1.731230915515877E+02 Tp= 2453186.891229265369
 N = 3.069820431806984E+00 MA= 1.666265057784200E+02 TA= 1.791733757787219E+02
 A = 3.088557852623704E-02 AD= 6.087194761192598E-02 PR= 1.172707029603336E+02

2453241.173611111 = A.D. 2004-Aug-23 16:10:00.0000 TDB
 EC= 9.644088659786864E-01 QR= 1.102706797220230E-03 IN= 2.695622940410984E+01
 OM= 2.074496159787979E+02 W = 1.733150689909747E+02 Tp= 2453186.529942960478
 N = 3.055408234165587E+00 MA= 1.669587136118652E+02 TA= 1.791044288580693E+02
 A = 3.098262608209883E-02 AD= 6.086254536697743E-02 PR= 1.178238626100691E+02

2453241.177083333 = A.D. 2004-Aug-23 16:15:00.0000 TDB
 EC= 9.571781880776633E-01 QR= 1.331431966378763E-03 IN= 2.733993431114083E+01
 OM= 2.073384489500795E+02 W = 1.734815088319407E+02 Tp= 2453186.132622670382
 N = 3.039244736141007E+00 MA= 1.672935873242254E+02 TA= 1.790376764568985E+02
 A = 3.109237808043960E-02 AD= 6.085332419450043E-02 PR= 1.184504807128825E+02

2453241.180555556 = A.D. 2004-Aug-23 16:20:00.0000 TDB
 EC= 9.491646445629344E-01 QR= 1.586854446641976E-03 IN= 2.766539748049446E+01
 OM= 2.072464180006995E+02 W = 1.736284947254189E+02 Tp= 2453185.696879748255
 N = 3.021271404362954E+00 MA= 1.676312431252539E+02 TA= 1.789731597816692E+02
 A = 3.121556705955376E-02 AD= 6.084427967246554E-02 PR= 1.191551343186619E+02

2453241.184027778 = A.D. 2004-Aug-23 16:25:00.0000 TDB
 EC= 9.403378222451394E-01 QR= 1.870588132747043E-03 IN= 2.794493864047313E+01
 OM= 2.071689725168481E+02 W = 1.737601423028330E+02 Tp= 2453185.220071714837
 N = 3.001428303261431E+00 MA= 1.679718016901806E+02 TA= 1.789109266663158E+02
 A = 3.135299788138643E-02 AD= 6.083540763002580E-02 PR= 1.199428950572681E+02

2453241.187500000 = A.D. 2004-Aug-23 16:30:00.0000 TDB
 EC= 9.306659260012543E-01 QR= 2.184408576316391E-03 IN= 2.818763652918707E+01
 OM= 2.071028986101721E+02 W = 1.738793228055684E+02 Tp= 2453184.699269219302
 N = 2.979654116040199E+00 MA= 1.683153893538546E+02 TA= 1.788510341799228E+02
 A = 3.150555636404568E-02 AD= 6.082670415177497E-02 PR= 1.208193924462685E+02

2453241.190972222 = A.D. 2004-Aug-23 16:35:00.0000 TDB
 EC= 9.201157005420302E-01 QR= 2.530272811529925E-03 IN= 2.840032768304010E+01
 OM= 2.070458618993433E+02 W = 1.739880866869438E+02 Tp= 2453184.131217710674
 N = 2.955886176065021E+00 MA= 1.686621395693124E+02 TA= 1.787935510970377E+02
 A = 3.167421919824429E-02 AD= 6.081816558495866E-02 PR= 1.217908872523788E+02

2453241.194444444 = A.D. 2004-Aug-23 16:40:00.0000 TDB
 EC= 9.086523468550708E-01 QR= 2.910342201115042E-03 IN= 2.858825769925238E+01
 OM= 2.069961259313743E+02 W = 1.740879222485917E+02 Tp= 2453183.512292796746
 N = 2.930060510443814E+00 MA= 1.690121947006971E+02 TA= 1.787385604872370E+02
 A = 3.186006537570906E-02 AD= 6.080978855030307E-02 PR= 1.228643567997410E+02

2453241.197916667 = A.D. 2004-Aug-23 16:45:00.0000 TDB
 EC= 8.962394328980501E-01 QR= 3.327008852365497E-03 IN= 2.875551816790810E+01
 OM= 2.069523722685787E+02 W = 1.741799189633929E+02 Tp= 2453182.838448044378
 N = 2.902111897903921E+00 MA= 1.693657082433181E+02 TA= 1.786861626237315E+02
 A = 3.206428940482315E-02 AD= 6.080156995728081E-02 PR= 1.240475945327999E+02

2453241.201388889 = A.D. 2004-Aug-23 16:50:00.0000 TDB
 EC= 8.957010852545865E-01 QR= 3.345201540451058E-03 IN= 2.876209011984316E+01
 OM= 2.069506628513128E+02 W = 1.741835851679324E+02 Tp= 2453182.809091593605
 N = 2.900900516590204E+00 MA= 1.693902453884949E+02 TA= 1.786852468743958E+02
 A = 3.207321522583879E-02 AD= 6.080122891122653E-02 PR= 1.240993953226475E+02

Table 6.21. Orbital ephemerides in 5 minute intervals for the orbit of *Cassini* around Saturn 2004 during OTM-2 on August 23, .

that over the seven weeks after the SOI, *Cassini*'s elliptical orbit became a little rounder, while the increasing periapsis distance and the fact that the apoapsis distance and semimajor axes more than doubled tell us that the orbit expanded. Seven and a half weeks after *Cassini*'s SOI, the spacecraft underwent OCM-2. The decreasing eccentricity data EC of Table 6.21 shows that *Cassini*'s

2453305.104166667 = A.D. 2004-Oct-26 14:30:00.0000 TDB	
EC= 9.049013829343483E-01 QR= 3.314678044313064E-03 IN= 2.881397385383604E+01	
OM= 2.073828454138249E+02 W = 1.738720925507729E+02 Tp= 2453306.898285608739	
N = 2.560624272711947E+00 MA= 3.554059354879828E+02 TA= 2.553165603201226E+02	
A = 3.485516558063887E-02 AD= 6.639565311696467E-02 PR= 1.405907160361037E+02	
2453305.111111111 = A.D. 2004-Oct-26 14:40:00.0000 TDB	
EC= 9.065299056253825E-01 QR= 3.306714420211525E-03 IN= 2.882717786153873E+01	
OM= 2.074550602198301E+02 W = 1.738601111638735E+02 Tp= 2453306.895770511590	
N = 2.504151333644243E+00 MA= 3.555309427818700E+02 TA= 2.554025583648885E+02	
A = 3.537724490743090E-02 AD= 6.744777539465027E-02 PR= 1.437612795853272E+02	
2453305.118055556 = A.D. 2004-Oct-26 14:50:00.0000 TDB	
EC= 9.087951739612423E-01 QR= 3.294402894123485E-03 IN= 2.884484466755378E+01	
OM= 2.075519611386019E+02 W = 1.738660351433712E+02 Tp= 2453306.892243521288	
N = 2.427214992631428E+00 MA= 3.556936643693552E+02 TA= 2.554496499168953E+02	
A = 3.612092733693199E-02 AD= 6.894745177974049E-02 PR= 1.483181346081385E+02	
2453305.125000000 = A.D. 2004-Oct-26 15:00:00.0000 TDB	
EC= 9.121103096243309E-01 QR= 3.273192536646883E-03 IN= 2.886917110592823E+01	
OM= 2.076855649056834E+02 W = 1.739316288736289E+02 Tp= 2453306.886975039262	
N = 2.318441577928716E+00 MA= 3.559149638097179E+02 TA= 2.554054630218427E+02	
A = 3.724205333590547E-02 AD= 7.121091413516406E-02 PR= 1.552767183901275E+02	
2453305.131944444 = A.D. 2004-Oct-26 15:10:00.0000 TDB	
EC= 9.172003620740877E-01 QR= 3.230099887268457E-03 IN= 2.890173988971150E+01	
OM= 2.078642812067753E+02 W = 1.742178083853473E+02 Tp= 2453306.878432036377	
N = 2.162545783999702E+00 MA= 3.562231406212043E+02 TA= 2.551014969930991E+02	
A = 3.901103879413930E-02 AD= 7.479197770101015E-02 PR= 1.664704639613075E+02	
2453305.138888889 = A.D. 2004-Oct-26 15:20:00.0000 TDB	
EC= 9.241798280179659E-01 QR= 3.118961850866344E-03 IN= 2.892208680644466E+01	
OM= 2.079757235815916E+02 W = 1.755268351030562E+02 Tp= 2453306.863977661356	
N = 1.997140583088844E+00 MA= 3.565547552036109E+02 TA= 2.538330999342962E+02	
A = 4.113630672857605E-02 AD= 7.915365160628576E-02 PR= 1.802577159807208E+02	
2453305.145833333 = A.D. 2004-Oct-26 15:30:00.0000 TDB	
EC= 9.202663941735291E-01 QR= 2.810064200676306E-03 IN= 2.869974794711623E+01	
OM= 2.067075160066805E+02 W = 1.816395929270097E+02 Tp= 2453306.850723981392	
N = 2.518455765018555E+00 MA= 3.557063083184860E+02 TA= 2.489662641502855E+02	
A = 3.524315966334228E-02 AD= 6.767625512600826E-02 PR= 1.429447382004534E+02	
2453305.152777778 = A.D. 2004-Oct-26 15:40:00.0000 TDB	
EC= 8.929505096960505E-01 QR= 2.528458027756747E-03 IN= 2.814937441425748E+01	
OM= 2.030950862034007E+02 W = 1.908276656849656E+02 Tp= 2453306.870836508926	
N = 4.590287273976390E+00 MA= 3.521136168721696E+02 TA= 2.430822942272282E+02	
A = 2.361952420864038E-02 AD= 4.471059038952401E-02 PR= 7.842646407795426E+01	
2453305.159722222 = A.D. 2004-Oct-26 15:50:00.0000 TDB	
EC= 8.770512570406397E-01 QR= 2.471928025651406E-03 IN= 2.791101005080209E+01	
OM= 2.012303228490764E+02 W = 1.943509179462299E+02 Tp= 2453306.892255879007	
N = 5.844925640581219E+00 MA= 3.498734696061307E+02 TA= 2.413282843843057E+02	
A = 2.010535420007086E-02 AD= 3.773878037449031E-02 PR= 6.159188707218551E+01	
2453305.166666667 = A.D. 2004-Oct-26 16:00:00.0000 TDB	
EC= 8.697144464614229E-01 QR= 2.465241478103078E-03 IN= 2.782392514556717E+01	
OM= 2.004795098669218E+02 W = 1.955671887853847E+02 Tp= 2453306.903915563133	
N = 6.401795217327748E+00 MA= 3.488784883223815E+02 TA= 2.408986773741027E+02	
A = 1.892183293655139E-02 AD= 3.537842439499971E-02 PR= 5.623422614731373E+01	

Table 6.22. Orbital ephemerides in 10 minute intervals for the orbit of *Cassini* during the Titan flyby of October 26, 2004.

trajectory became steadily rounder. The increasing periapsis distances QR along with the stable apoapsis distances AD confirm that its orbit continued to expand. The flyby of Saturn's moon Titan had an appreciable effect on *Cassini's* elliptical orbit. The data of Table 6.22 inform us that the eccentricities EC decreased, rounding the ellipse further. It also tightened the orbit by decreasing the periapsis distance QR by nearly one-third and the apoapsis distance AD by close to half.

During the entire time from *Cassini's* approach to Saturn on July 1st, 2004 until its departure from the gravitational influence of Titan on October 26th, 2004, the inclination IN of *Cassini's* orbital plane relative to that of Earth increased slowly, only to decrease again towards the end of the passage of Titan. The angles measuring the orientation OM of the nodal line and the orientation W of the focal axis of the trajectory were very stable throughout. But toward the end of *Cassini's* swing around Titan, OM decreased and W increased. All in all, the data in the tables that describe *Cassini's* three important maneuvers are aligned with the changing orbital geometry that diagram of Figure 6.24 depicts. Notice that these three maneuvers and the evolving modifications in the craft's trajectory that they effected all occurred during the craft's critical first loop around Saturn.

Problem 6.23. Consider the parabola of Figure 6.34 with its focal point $(d, 0)$ and its directrix the line $x = -d$. By definition, a point $P = (x, y)$ is on the parabola, if the distance from (x, y) to $(d, 0)$ is equal to the distance from (x, y) to the line vertical line through $x = -d$. So $P = (x, y)$ is on the parabola precisely if $\sqrt{(x-d)^2 + y^2} = x + d$. After squaring both sides, $x^2 - 2dx - d^2 + y^2 = x^2 + 2dx + d^2$. So $P = (x, y)$ is on the parabola, precisely if $y^2 = 4dx$.

The equation of the upper half of the parabola is $y = 2\sqrt{dx} = 2(dx)^{\frac{1}{2}}$. A look at the figure tells us that the area B is the difference between a parabolic section and a triangle. In particular,

$$B = \int_0^x 2(dx)^{\frac{1}{2}} dx - \frac{1}{2}xy.$$

There is a lesson to be drawn from this conclusion. What was just derived is problematic for two reasons. The first is that x serves simultaneously in two different roles. In addition to being the x coordinate of P , it is also the variable of integration. The second problem is the choice of the constant d in the definition of the parabola, and the consequence that the two dx in the integrand have two totally different meanings. In response, let's write the constant d as c and the point P as (x_1, y_1) . We now get

$$B = \int_0^{x_1} 2c^{\frac{1}{2}}x^{\frac{1}{2}} dx - \frac{1}{2}x_1y_1 = 2\sqrt{c}\left(\frac{2}{3}x^{\frac{3}{2}}\Big|_0^{x_1}\right) - \frac{1}{2}x_1(2\sqrt{cx_1}^{\frac{1}{2}}) = \frac{4}{3}\sqrt{cx_1}^{\frac{3}{2}} - \sqrt{cx_1}^{\frac{3}{2}} = \frac{1}{3}\sqrt{cx_1}^{\frac{3}{2}}.$$

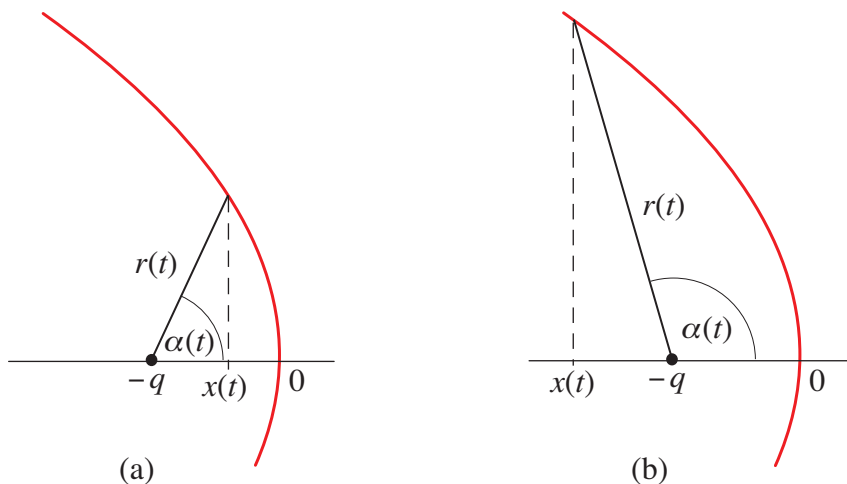
We can now translate back to get that the area B of Figure 6.34 is equal to $\frac{1}{3}\sqrt{dx}^{\frac{3}{2}}$.

Turn to the paragraph *Parabolic Trajectories*. The discussion in the paragraph is complete, but we'll fill in a few details. Focus on the parabolic trajectory of C with focal point S and its motion along it as illustrated in Figure 6.35. Review the meaning of the time variable t , angle $\alpha(t)$, and the function $A(t)$. Recall that $\alpha(t) \geq 0$ when $t \geq 0$, and $\alpha(t) < 0$ when $t < 0$. For $t \geq 0$, $A(t)$ is the area swept out by the segment SC from $t = 0$ to t . For a negative t , $A(t)$ is minus the area swept out from t to $t = 0$. With q the periapsis distance of the trajectory of C , the focal point is positioned at $(0, -q)$. The directrix is the line $x = q$ and (by applying the first part of Problem 6.23) the equation of the parabola is

$$y^2 = -4qx.$$

Since $x \leq 0$, the two solutions of $y^2 = -4qx$ for y are $y = +2\sqrt{q}(-x)^{\frac{1}{2}}$ and $y = -2\sqrt{q}(-x)^{\frac{1}{2}}$. The first is the equation for the upper half of the parabola and the second is the equation for the lower half. Another look at Figure 6.35 informs us that with $x = -q$, the length $2y = 4\sqrt{q}(-q)^{\frac{1}{2}} = 4q$ is the latus rectum of the parabola.

Recall that the coordinates of the position C of the craft at time t are $x(t)$ and $y(t)$, so that $C = (x(t), y(t))$, and that $r(t)$ is the distance from S to C . Suppose $0 \leq \alpha(t) \leq \frac{\pi}{2}$ and refer to diagram (a) of Sol 6.5. Since $x(t) \leq 0$, $\cos \alpha(t) = \frac{q+x(t)}{r(t)}$. If $\frac{\pi}{2} \leq \alpha(t) \leq \pi$, then by diagram (b), $\cos(\pi - \alpha(t)) = \frac{-x(t)-q}{r(t)}$. Since $\cos(\pi - \alpha(t)) = -\cos \alpha(t)$, we get $\cos \alpha(t) = \frac{q+x(t)}{r(t)}$ in this case also.



Sol 6.5

Since $\cos \alpha(t) = \cos(-\alpha(t))$, this equality holds for negative $\alpha(t)$ as well. Recall that $\beta(t)$ is defined by $\beta(t) = \tan \frac{\alpha(t)}{2}$ and that $\beta(t) = \pm \frac{(-x(t))^{\frac{1}{2}}}{\sqrt{q}}$ with the $+$ in place when $t \geq 0$, and the $-$ when $t \leq 0$.

The paragraph goes on to establish the parabolic version

$$\beta(t) + \frac{1}{3}\beta(t)^3 = \frac{\kappa t}{q^2} = \sqrt{\frac{GM}{2q^3}} t$$

of Kepler's equation, where κ is the Kepler constant of the trajectory and GM is the gravitational constant of the attracting body. The solution of Kepler's equation for $\beta(t)$ in terms of t is complicated, but it can be cast in an explicit form (unlike those of the elliptical situation of Chapter 5E and the hyperbolic situation in section 6H).

Problem 6.25. For $f(x) = x(x + 3q)^2 - \frac{9}{2}GMt^2$ and t a constant, we get

$$f'(x) = (x + 3q)^2 + 2x(x + 3q) = (x + 3q)(x + 3q + 2x) = 3(x + 3q)(x + q) \text{ and}$$

$$f''(x) = 3(x + q) + 3(x + 3q) = 6(x + 2q)$$

by applying both the chain and product rules. The second derivative test tells us the following: Since $f'(-3q) = 0$ and $f''(-3q) = -6q < 0$, $y = f(x)$ has a local maximum at $x = -3q$; since $f'(-q) = 0$ and $f''(-q) = 6q > 0$, $y = f(x)$ has a local minimum at $x = -q$; and since $f''(-2q) = 0$ with $f''(x) < 0$ for $x < -2q$ and $f''(x) > 0$ for $x > -2q$, $y = f(x)$ has a point of inflection at

$x = -2q$. Notice that $f(-3q) = -\frac{9}{2}GMt^2$ and $f(-q) = -4q^3 - \frac{9}{2}GMt^2$. For $x > -q$, $f'(x) = 3(x+3q)(x+q) > 0$ and $f''(x) = 6(x+2q) > 6q > 0$. Therefore $y = f(x)$ is increasing and concave up for $x > -q$, and hence for $x \geq 0$. Because $f(0) = -\frac{9}{2}GMt^2$, it follows that $y = f(x)$ has a unique positive real root.

To see that $q\beta(t)^2$ is the positive real root of $f(x) = x(x+3q)^2 - \frac{9}{2}GMt^2$ requires details. Return to Kepler's equation and rewrite it as $\sqrt{q^3}\beta(t)(1+\frac{1}{3}\beta(t)^2) = \frac{1}{\sqrt{2}}\sqrt{GM}\cdot t$. After multiplying through by 3, we get $\sqrt{q^3}\beta(t)(3+\beta(t)^2) = \frac{3}{\sqrt{2}}\sqrt{GM}\cdot t$. By squaring both sides, $q^3\beta(t)^2(3+\beta(t)^2)^2 = \frac{9}{2}GMt^2$. Finally, after rewriting things once more,

$$q\beta(t)^2(3q + q\beta(t)^2)^2 = \frac{9}{2}GMt^2,$$

So $q\beta(t)^2$ is the unique positive real root of $f(x) = x(x+3q)^2 - \frac{9}{2}GMt^2$.

The rest of the discussion of the *Parabolic Trajectories* in text is complete. This includes the formula

$$\beta(t) = \pm \sqrt{\frac{1}{q}\left(\frac{9}{4}GMt^2 + z(t) + q^3\right)^{\frac{1}{3}} + \frac{1}{q}\left(\frac{9}{4}GMt^2 - z(t) + q^3\right)^{\frac{1}{3}} - 2}$$

for $\beta(t)$, with $z(t) = \sqrt{(\frac{9}{4}GMt^2 + q^3)^2 - q^6}$, where the $+$ applies if $t \geq 0$, and the $-$ if $t < 0$. It also includes the parabolic velocity formulas.

We will now apply these results to the study of Lovejoy's comet C/2011 W3. In reference to its trajectory, the perihelion distance was 0.00555381 au. With 1 au = 149597870.7 km, we'll take $q = 8.3083815 \times 10^5$ km. Since its eccentricity of $\varepsilon = 0.99992942$ was very close to 1, we will assume that the comet's trajectory was parabolic.

Problem 6.27. Since seven days is equal to $7(86,000) = 6.04800 \times 10^5$ seconds, the comet was $t = 6.04800 \times 10^5$ sec past perihelion. The assumption that the comet's orbit is parabolic along with the data for the orbit that was supplied, calls for the solution of the parabolic Kepler equation

$$\beta(t) + \frac{1}{3}\beta(t)^3 = \sqrt{\frac{GM}{2q^3}} t = \sqrt{\frac{1.3271244 \times 10^{11}}{2(8.3083815 \times 10^5)^3}} (6.04800 \times 10^5) = 205.7206826$$

for $\beta(t)$. Since $t \geq 0$,

$$\beta(t) = \sqrt{\frac{1}{q}\left(\frac{9}{4}GMt^2 + z(t) + q^3\right)^{\frac{1}{3}} + \frac{1}{q}\left(\frac{9}{4}GMt^2 - z(t) + q^3\right)^{\frac{1}{3}} - 2},$$

where $z(t) = \sqrt{(\frac{9}{4}GMt^2 + q^3)^2 - q^6}$. Computing $z(t)$, we get

$$\begin{aligned} z(t) &= \sqrt{\left[\frac{9}{4}(1.3271244 \times 10^{11})(6.04800 \times 10^5)^2 + (8.3083815 \times 10^5)^3\right]^2 - (8.3083815 \times 10^5)^6} \\ &= 10.92244830 \times 10^{22}. \end{aligned}$$

It follows that

$$\begin{aligned} &\left(\frac{9}{4}GMt^2 + z(t) + q^3\right)^{\frac{1}{3}} \\ &= \left(\frac{9}{4}(1.3271244 \times 10^{11})(6.04800 \times 10^5)^2 + (10.92244830 \times 10^{22}) + (8.3083815 \times 10^5)^3\right)^{\frac{1}{3}} \end{aligned}$$

$$\begin{aligned}
&= 6.02259045 \times 10^7, \text{ and} \\
&\left(\frac{9}{4}GMt^2 - z(t) + q^3\right)^{\frac{1}{3}} \\
&= \left(\frac{9}{4}(1.3271244 \times 10^{11})(6.04800 \times 10^5)^2 - (10.92244830 \times 10^{22}) + (8.3083815 \times 10^5)^3\right)^{\frac{1}{3}} \\
&= -0.00352381 \times 10^7.
\end{aligned}$$

By substituting into the formula for $\beta(t)$, we get

$$\begin{aligned}
\beta(t) &= \sqrt{\frac{1}{(8.3083815 \times 10^5)}(6.02259045 \times 10^7) + \frac{1}{(8.3083815 \times 10^5)}(-0.00352381 \times 10^7) - 2} \\
&= 8.39319475.
\end{aligned}$$

Plugging the value $\beta(t) = 8.39319475$ into the formula $r(t) = q(\beta(t)^2 + 1)$ and the result into

$$v(t) = \sqrt{GM} \sqrt{\frac{2}{r(t)}} \quad \text{and} \quad \gamma(t) = \pi - \sin^{-1} \frac{\sqrt{GM} q(1+\varepsilon)}{r(t)v(t)},$$

we get the following information about the trajectory of Lovejoy's comet at time t ,

$$\begin{aligned}
r(t) &= q(\beta(t)^2 + 1) = (8.3083815 \times 10^5)(8.39319475^2 + 1) \approx 59,359,828 \text{ km}, \\
v(t) &= \sqrt{GM} \sqrt{\frac{2}{r(t)}} = \sqrt{1.3271244 \times 10^{11}} \sqrt{\frac{2}{59,359,828}} = 66.86894909 \approx 66.87 \text{ km/sec, and} \\
\gamma(t) &= \pi - \sin^{-1} \frac{\sqrt{2GMq}}{r(t)v(t)} = \pi - \sin^{-1} \frac{\sqrt{2(1.3271244 \times 10^{11})(8.3083815 \times 10^5)}}{(5.93598283 \times 10^7)(66.86894909)} = \pi - 0.11858513 \text{ radians} \\
&\approx 180^\circ - 6.7944^\circ \approx 173.21^\circ.
\end{aligned}$$

Problem 6.29. Since $\frac{v(t_0)}{\sqrt{GM}}$ is a positive constant and the function $f(x) = \sqrt{x}$, $0 \leq x < \infty$, is a continuous function, there is a positive number x_0 such that $\sqrt{x_0} = \frac{v(t_0)}{\sqrt{GM}}$ and hence $v(t_0) = \sqrt{GM}\sqrt{x_0}$. If this were not so, the horizontal line $y = \frac{v(t_0)}{\sqrt{GM}}$ would not intersect the graph of $f(x) = \sqrt{x}$. This would mean that the graph of the function, see Figure 6.36, would have at least two disjoint components, thus violating its continuity.

If $v(t_0) < \sqrt{GM}\sqrt{\frac{2}{r(t_0)}}$, then $\sqrt{GM}\sqrt{x_0} < \sqrt{GM}\sqrt{\frac{2}{r(t_0)}}$. Since $g(x) = x^2$ is an increasing function, $x_0 < \frac{2}{r(t_0)}$. Set $\frac{2}{r(t_0)} - x_0 = D$. Note that D is positive and that $v(t_0) = \sqrt{GM}\sqrt{x_0} = \sqrt{GM}\sqrt{\frac{2}{r(t_0)} - D}$.

If $v(t_0) > \sqrt{GM}\sqrt{\frac{2}{r(t_0)}}$, then $\sqrt{GM}\sqrt{x_0} > \sqrt{GM}\sqrt{\frac{2}{r(t_0)}}$, so that $x_0 > \frac{2}{r(t_0)}$. In this case, set $D = x_0 - \frac{2}{r(t_0)}$. Again D is positive, and this time $v(t_0) = \sqrt{GM}\sqrt{x_0} = \sqrt{GM}\sqrt{\frac{2}{r(t_0)} + D}$.

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